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When the Rules of Discourse Change, but Nobody Tells You: Making Sense of Mathematics Learning From a Commognitive Standpoint

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The interpretive framework for the study of learning introduced in this article and called *commognitive* is grounded in the assumption that thinking is a form of communication and that learning mathematics is tantamount to modifying and extending one's discourse. These basic tenets lead to the conclusion that substantial discursive change, rather than being necessitated by an extradiscursive reality, is spurred by *commognitive conflict*, that is, by the situation that arises whenever different interlocutors are acting according to differing discursive rules. The framework is applied in 2 studies, one of them featuring a class learning about negative numbers and the other focusing on 2 first graders learning about triangles and quadrilaterals. In both cases, the analysis of data is guided by questions about (a) features of the new mathematical discourse that set it apart from the mathematical discourse in which the students were conversant when the learning began; (b) students' and teachers' efforts toward the necessary discursive transformation; and (c) effects of the learning–teaching process, that is, the extent of discursive change actually resulting from these efforts. One of the claims corroborated by the findings is that school learning requires an active lead of an experienced interlocutor and needs to be fueled by a learning-teaching agreement between the interlocutor and the learners.

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The aim of this article is to propose an interpretive framework for making sense of classroom processes. This offering comes as an answer to the deeply felt need for methods of analysis, the sensitivity of which would match the intricacy of the available data. These days, research on learning is in a state of perturbation. Ever since audio and video recorders have become standard tools of the researcher's trade, our ability to interpret human activities lags behind our ability to observe and to see. In this respect, our current situation is comparable to that of the 17th-century scientists just faced with the newly invented microscope: Powerful, high-resolution lenses that reveal what was never noticed before are yet to be matched by an equally powerful theoretical apparatus. The available approaches to the study of learning, whether traditional or novel, still leave quite a lot to wish for. Under recurrent scrutiny, the durable, high-resolution replicas of real-life events prove too complex and too full of fine details to yield to the rather blunt tools of the traditional cognitivist approaches. The newer perspectives, in contrast, are not yet accompanied by broadly applicable methods of study. Our helplessness as researchers is aggravated by the fact that the current reform, promoting the pedagogy of talking classrooms and of communities of inquiry, makes learning processes not only more visible, but also much more intricate and messy.

Unlike in the past, when coarse-grained manually collected data supported claims on cross-situational invariants and drew attention to the results rather than the process of learning, researchers are now attracted by change and diversity. In order to fully capitalize on the increased visibility, and thus investigability, of human processes, we need an analytic lens that extends our field of vision so as to include both the "how" and the "what" of learning. The final test of such approaches will be their ability to support empirical studies on learning and to give some truly insightful advice to those who try to improve teaching and learning. If an interpretive framework is to pass the test, studies guided by this framework must be able to cope with the following questions:

1. *Focus on the object of learning:* In the case under study, what kind of change was supposed to occur as a result of learning?
2. *Focus on the process:* How did the students and the teacher work toward this change?
3. *Focus on the outcome:* Has the expected change occurred?

Many researchers are working these days toward conceptual frameworks that would help in meeting the challenge of these newly discovered complexities. Some promising proposals have already been made, and some others are on their way. The approach introduced in this article is the result of years-long efforts to fathom the intricacies of mathematical learning. Although tailored to the special needs of the particular school subject, this approach is believed to be applicable to almost any other subject. In what follows, I begin with the fundamentals of a *commognitive* framework, as this approach will be called, and I continue by applying this framework in two empirical studies, one featuring a seventh-grade class learning about negative numbers and the other focused on two first graders learning about triangles and quadrilaterals. The commognitive analyses of the two corpora of data are guided by the three questions listed above.

A COMMOGNITIVE APPROACH TO THE STUDY OF LEARNING

In the following paragraphs, the introduction to the commognitive approach is preceded by a concise account of the developments that gave rise to this framework and is followed by an attempt to situate this perspective among several other comparable discourses on learning.[1]

The Origins of a Commognitive Approach: Transition From Acquisitionist to Participationist Perspectives

It seems that in order to find what we are looking for, we need to revise our discourse on learning. Traditional educational studies conceptualize learning as the "acquisition" of entities such as ideas or concepts.[2] Due to the crudeness of these atomic units, those who work within the acquisitionist framework are compelled to gloss over fine details of messy interpersonal interaction within which the individual acquisition takes place. If the acquisitionist researchers notice these particularities at all, they quickly dismiss them as mere noise. Overgeneralizations and unwarranted statements inevitably follow, thus diminishing the value of this kind of study as a basis on which to build better pedagogies.

The disillusionment with acquisitionism, although greatly precipitated by the advent of digital recording, began, in fact, prior to the advances in data-collecting techniques. Cross-cultural and cross-situational studies that had proliferated since the first decades of the 20th century systematically undermined acquisitionist claims about developmental invariants. Their results drew researchers' attention to the social and cultural contexts of learning. The resulting shift of emphases was not a mere quantitative change. The diverse areas of research dealing with those forms

[1]To complete the commognitive definition of human thinking, let me add that within this framework, communication is defined as a collectively performed rules-driven activity that mediates and coordinates other activities of the collective. For justification and elaboration of this definition, see Sfard (in press).

[2]In this context, it does not matter whether the word *acquisition* is interpreted as passive reception or as active construction.

of life that can be found only in humans[3] now promote, either explicitly or as an inevitable entailment, the participationist vision of the origins of the human uniqueness. Rather than inquiring about personal acquisitions, participationists conceptualize developmental transformations as changes in what and how people are doing and claim that patterned collective activities are developmentally prior to those of the individual. Although certainly situated at the crossroads of several traditions, this vision of human development is usually traced back to the work of Vygotsky and other founders of activity theory (see, e.g., Engeström, 1987; Leontiev, 1947/1981).

The reasons and consequences of the transition from the acquisitionist to participationist understanding of human development deserve additional attention. The long-standing acquisitionist tenets, although initially quite productive, prove unhelpful not only when it comes to accounting for interpersonal and cross-situational differences, but also when one tries to fathom the sources of those changes in human ways of acting that transcend a single life span. Within the confines of acquisitionist discourse, which views human development as shaped every time anew by the same, basically immutable factors, there is no cogent explanation for the fact that human forms of life, unlike those of other species, evolve over history and that the outcomes of the ongoing transformations accumulate from generation to generation, constantly redefining the nature and extent of individual growth.[4]

The participationist account comes to the rescue not only by offering a different answer to the question of how humans develop, but also by altering the conception of what it is that develops. When speaking about human learning, participationists do not mean transformations in individuals, but rather in what and how people are doing—in patterned human processes, both individual and collective. This change means a different unit of analysis. Among eligible candidates one should count *form of life*, suggested by Wittgenstein (1953), and *activity*, the pivotal idea of the activity theory. The current popular term *practice* is yet another viable option (see, e.g., Cobb, 2002; Wenger, 1998). My noncommittal expression *patterned human processes* (or forms of doing) embraces, basically, the same idea. Whatever name and definition are chosen and whatever claims about humans are formulated with their help, the strength of this special construct is in the fact that it has both collective and individual "editions," and that it pictures the human society as a huge

[3]For example, psychology, sociology, anthropology, cultural studies.

[4]The route acquisitionists usually take around this dilemma is grounded in the claim about reflexivity of the relation between activity and genes: Whereas genes have an impact on human doings, the way people do things can modify the genetic blueprints in return. Although empirically corroborated and of significance, this claim, per se, does not explain why only humans seem capable of such genetic accumulation. Also in this case, to arrive at a more satisfactory account one needs to extend the unit of analysis beyond the single individual. Therefore, the genetic explanation can perhaps complement the participationist account, but it cannot replace it.

fractal-like entity, every part of which is a society in itself, indistinguishable in its inner structure from the whole.

Armed with this flexible analytic focus, participationists view the ongoing transformations in human forms of doing as the result of two complementary processes, that of *individualization of the collective* and that of *communalization of the individual.*[5] Individualization results in personal versions of collective activities: learning to speak, to solve mathematical problems, or to cook means a gradual transition from being able to take a part in collective implementations of a task to becoming capable of implementing such a task in its entirety on one's own accord. Eventually, a person can perform in his or her unique way entire sequences of steps that, until that point, he or she would have only executed with others. The processes of individualization and communalization are reflexively interrelated: The collective activities are primary models for individual forms of acting, whereas individual variations feed back into the collective forms of doing, acquire permanence, and are carried in space and time from one community of actors to another. This reconceptualization of human development resolves, therefore, the puzzle of the historical change in human forms of doing. The participationist perspective on uniquely human forms of activity, if taken seriously, is bound to bring about a reconceptualization of such basic terms as *thinking*, *mathematics*, and *learning*.

Basic Commognitive Tenets

Thinking as individualization of (interpersonal) communication. Although thinking appears to be an inherently individual activity, there is no reason to assume that its origins are different from those of any other uniquely human capacity: Like all the others, this special form of human doing is most likely to have developed from a patterned collective activity. At close look, the best candidate for the collective activity that morphed into thinking through the process of individualization is interpersonal communication. It seems, therefore, that human thinking can be regarded (defined, in fact) as the individualized form of the activity of communicating, that is, as communication with oneself. This self-communication does not have to be in any way audible or visible, and it does not have to be in words. Additional support for this definition comes from the fact that the phenomena we usually label with the name *thinking* are clearly dialogical in nature—they are

[5]The term *individualization* refers to the process of gradual overtaking of the roles of others, accompanied by an enhancement of one's agency over the given activity. It may be viewed as a version of what Vygotsky called *internalization* and what Bakhtin and Leont'ev renamed *appropriation* so as to capture both the active nature and the bidirectionality of the process (Cazden, 2001, p. 76). In my work I have opted for the term *individualization* as one that is free of objectifying, acquisitionist undertones of both *internalization* and *appropriation* and also implies the inevitability of personal variations.

acts of informing ourselves, arguing, asking questions, and waiting for our own responses.[6]

In spite of all this argumentation, the claim that thinking originates in interpersonal communicating may be difficult to accept. After all, whatever we call thinking is usually done by each one of us alone and is generally considered to be inaccessible to others in the direct manner. More than any other form of human doing, this individually performed activity appears to grow from inside the person and be biologically determined. Another related obstacle to treating thinking as a type of communication is that the thinking–communicating dichotomy is entrenched in both our everyday and scientific discourses. Indeed, our speaking about thoughts as being "conveyed" or "expressed" in the act of communication implies two distinct processes, that of thinking and that of communicating, with the former slightly preceding the latter and constantly feeding into it. According to this vision, the outcomes of thinking, pictured as entities in their own right, are supposed to preserve their identity while being "put in other words" or "expressed somehow differently."

Whereas acquisitionists have been working with this dualist vision of human cognition for centuries, participationists are likely to view the idea of thought conveyed in communication as but a direct result of an unhelpful objectification. With Wittgenstein (1953), they believe that "Thought is not an incorporeal process which lends life and sense to speaking, and which it would be possible to detach from speaking" (p. 108). Having accepted this claim, one can also see that it remains in force when the somewhat limiting word *speaking* is replaced with the more general term *communicating*. Consequently, thinking stops being a self-sustained process separate from and, in a sense, primary to any act of communication and becomes an act of communication in itself, although not necessarily interpersonal. To stress this fact, I propose to combine the terms *cognitive* and *communicational* into the new adjective *commognitive*. The etymology of this new word will always remind us that whatever is said with its help refers to phenomena traditionally included in the term *cognition*, as well as to those usually associated with interpersonal exchanges.

Mathematics as a form of discourse. With its roots in a patterned collective activity, commognition—both thinking and interpersonal communication—must follow certain rules. These rules are not anything the participants would observe in a conscious way, nor are they in any sense natural or necessary. The source of the rules is in historically established customs. This contingent nature of communicational patterns is probably the reason why Wittgenstein (1953) decided to

[6]Note that this definition does not assume any special mode of communication and, in particular, makes no reference to language. From here it follows that the claim "thinking is a form of communication" should not be confused with the controversial assertion "thinking does not exist without language," which has been stirring fierce debates for centuries.

speak about communication as a kind of game.[7] Just as there is a multitude of games, played with diverse tools and according to a multitude of rules, so there are many types of commognition, differing one from another not only in their rules, but also in the objects they refer to and in the media they use. Like in the case of games, individuals may be able to participate in certain types of communicational activity and be unable to take part in some others. The different types of communication that bring some people together while excluding some others are called *discourses*. Given this definition, any human society may be divided into partially overlapping *communities of discourses*. To be members of the same discourse community, individuals do not have to face one another and do not need to actually communicate. The membership in the wider community of discourse is won through participation in communicational activities of any collective that practices this discourse, be this collective as small as it may.

The focus of this article is on one particular type of discourse (thus thinking), called *mathematical.* Mathematical discourse is made distinct by a number of interrelated features.

A discourse counts as mathematical if it features mathematical words, such as those related to quantities and shapes. While becoming a participant of school mathematical discourse, a student learns new uses of previously encountered words, such as *triangle* or *square*, but may also have to learn terms that he or she has never used before and that are unique to mathematics. Expressions such as *negative 2* (or *minus 2*) or *negative half* (*minus half*) are good examples of the latter type of lexical innovation. Although shape- and number-related words may appear in nonspecialized, colloquial discourses, literate mathematical discourses as practiced in schools or in academia dictate their own more disciplined uses of these words. Word use is an all-important matter because, being tantamount to what others call "word meaning" ("The meaning of a word is its use in language," Wittgenstein, 1953, p. 20), it is responsible to a great extent for how the user sees the world.

Visual mediators are means with which participants of discourses identify the object of their talk and coordinate their communication. Whereas colloquial discourses are usually mediated by images of independently existing material things (i.e., of concrete objects that are pointed to with nouns or pronouns and that may be either actually seen or just imagined), mathematical discourses often involve symbolic artifacts, created specially for the sake of this particular form of communication. The most common examples include mathematical formulae, graphs, drawings, and diagrams. While communicating, we attend to the mediators in special ways. Think, for example, about the extended number line and the way you scan it

[7]More precisely, Wittgenstein (1953) spoke about *language* games. The metaphor of game, however, is clearly applicable also to nonverbal forms of communication.

with your eyes while trying to add two numbers. Contrary to what is implied by the common understanding of the role of tools, within the communicational framework one does not conceive of artifacts used in communication as mere auxiliary means for conveying or giving expression to preexisting thought. Rather, one views them as a part and parcel of the act of communication and thus, in particular, of thinking processes.

Narrative is any text, spoken or written, that is framed as a description of objects, or of relations between objects or activities with or by objects, and that is subject to *endorsement* or rejection, that is, to being labeled as *true* or *false*. Terms and criteria of endorsement may vary considerably from discourse to discourse, and, more often than not, the issues of power relations between interlocutors play a considerable role. This is certainly true about social science and humanistic narratives such as history or sociological theories. Mathematical discourse has been conceived as one that should be impervious to any considerations other than purely deductive relations between narratives (clearly, the reality may be quite different). In the case of scholarly mathematical discourse, the consensually endorsed narratives are known as mathematical *theories*, and this includes such discursive constructs as *definitions*, *proofs*, and *theorems*. One can divide mathematical narratives into *object level*, that is, stories about mathematical objects (e.g., $2 + 3 = 5$, $(a + b)^2 = a^2 + 2ab + b^2$, "the sum of the angles in a triangle is 180^o"); and *meta level*, that is, stories about the discourse itself (and this includes narratives about how mathematics is done; e.g., "While calculating, perform the operation in brackets first" the heuristic rules of proving).

Routines are well-defined repetitive patterns in interlocutors' actions, characteristic of a given discourse. Specifically mathematical regularities can be noticed whether one is watching the use of mathematical words and mediators or following the process of creating and substantiating narratives about numbers or geometrical shapes. In fact, such repetitive patterns can be seen in almost any aspect of mathematical discourses: in mathematical forms of categorizing, in mathematical modes of attending to the environment, and in ways of viewing situations as the same or different, which is crucial for the interlocutors' ability to apply mathematical discourse whenever appropriate. The list is longer still. Thus, the routine is an all-encompassing category that partially overlaps with the three former characteristics (word use, mediator use, and endorsing narratives) but is much broader than that. Some of the routines implemented by participants of mathematical discourse are dictated by properties of mathematical objects that are being manipulated. Think, for example, about the routines of numerical calculations that are grounded in the properties of associativity, commutativity, and distributivity of addition and multiplication. Principles that regulate this kind of routine are usually explicit and are called *object-level rules*. There are other rules of which the interlocutors are much less aware, but that nevertheless can be deduced from their actions. This other rules are called *meta level*, or simply *meta-rules*, because, if formulated, they would take the form of metalevel narratives—propositions about the discourse rather than about its objects. Note that in most cases, the rules do not determine the routine course of action but only constrain it and give it general direction. Meta-rules, such as those that govern word use or those that regulate the endorsement of mathematical narratives (i.e., the rules of proving or defining), are rarely made explicit and are usually learned from examples rather than from general verbal prescriptions. It must be emphasized that there is more than one type of communication that can count as mathematical, and that some mathematical routines that are acceptable in a school (e.g., school routines for endorsement of narratives) would be deemed inappropriate if applied in scholarly mathematical research.[8]

Learning mathematics as changing a discourse. Learning mathematics may now be defined as individualizing mathematical discourse, that is, as the process of becoming able to have mathematical communication not only with others, but also with oneself. Because mathematical discourse learned in school is a modification of children's everyday discourses, learning mathematics may be seen as transforming these spontaneously learned colloquial discourses rather than as building new ones from scratch. In particular, a person who learns about negative numbers or geometric figures alters and extends his or her discursive skills so as to become able to use this form of communication in solving mathematical problems.

Within the commognitive framework, therefore, asking what the participants of a study have yet to learn becomes equivalent to inquiring about required changes in students' ways of communicating. Discursive development of individuals or of entire classes can then be studied by identifying transformations in each of the four discursive characteristics: the use of words characteristic of the discourse, the use of mediators, endorsed narratives, and routines.

Commognitive conflict as a source of mathematical learning. If learning mathematics is a change of discourse, one can distinguish between two types of learning: *object-level learning*, which expresses itself in the expansion of the existing discourse, attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; and *metalevel learning*, which involves changes in meta-rules of the discourse. This latter change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way. Considering the contin-

[8]The commognitive use of the term *routine* is close to the usage that was proposed by Schutz and Luckmann (1973) and applied in the context of mathematics learning by Voigt (1985).

gency of metadiscursive rules—the fact that these rules are a matter of a useful custom rather than of necessity—it is rather implausible that learners would initiate a metalevel change by themselves. The metalevel learning is most likely to originate in the learner's direct encounter with the new discourse. Because this new discourse is governed by meta-rules different from those according to which the student has been acting so far, such an encounter entails *commognitive conflict*—a situation in which communication is hindered by the fact that different discursants are acting according to different meta-rules (and thus possibly using the same words in differing ways).

Usually, the differences in meta-rules that are the source of the conflict find their explicit, most salient expression in the fact that different participants endorse contradicting narratives. Of course, some cases of conflicting narratives may stem from differing opinions rather than from discursive conflict. Discursive conflict should be suspected only in those cases when the conflicting narratives are factual (i.e., endorsable according to well-defined metadiscursive rules) and the possibility of an error in their construction and substantiation has been eliminated. As simple as this last claim may sound, the presence of commognitive conflict is not easy to detect. Only too often, commognitive conflicts are mistaken for factual disagreements, that is, as a clash between two sentences only one of which can be correct.[9]

The notion of commognitive conflict should not be confused with the acquisitionist idea of *cognitive conflict*, central to the well-known, well-developed theory of conceptual change (Schnotz, Vosniadou, & Carretero, 1999; Vosniadou, 1994). At least three substantial differences can be listed. First, acquisitionists and commognitivists do not agree about the locus of the conflict. Cognitive conflict is defined as arising in the encounter between one's beliefs and the world: A person holds two contradicting beliefs about the world, with one of these beliefs being, of necessity, incompatible with the real state of affairs. In one's attempt to resolve the conflict, the person will try to employ the world itself as an ultimate arbitrator. The idea of commognitive conflict, in contrast, rests on the assumption that learning, as a change of discourse, is most likely to result from interactions with others. According to this latter approach, the main opportunities for metalevel learning arise not from discrepancies between one's endorsed narratives and certain external evidence, but from differences in interlocutors' ways of communicating. The com-

[9]The majority of the well-known incompatibilities between scientific theories may, in fact, result from commognitive conflicts rather than from correct versus incorrect factual beliefs. Thus, for example, what appears as a straightforward contradiction between Aristotle and Newton—between the former thinker's claim that a constant force applied to a body results in the body's constant movement and Newton's second law of dynamics asserting that constant force results in a constant acceleration—may, in fact, be the outcome of the two men's differing uses of the word *force*.

mognitive framework, therefore, questions the traditional relation between the world and the discourse: Rather than assuming that what we say (think) about the world is determined by what we find in the world, it claims a reflexive relation between what we are able to say and what we are able to perceive and endorse. Most of the time, our discourses remain fully consistent with our experience of reality. We need a discursive change to become aware of new possibilities and arrive at a new vision of things. We thus often need a change in how we talk before we can experience a change in what we see.

The second difference between the two types of conflict is in their significance for learning: Whereas creating cognitive conflict is considered to be an optional pedagogical move, particularly useful when students display "misconceptions," the commognitive conflict is the most likely, often indispensable, source of metalevel mathematical learning. Without other people's example, children may have no incentive for changing their discursive ways. From the children's point of view, the discourse in which they are fluent does not seem to have any particular weaknesses as a tool for making sense of the world around them.

Finally, the commognitive and acquisitionist versions of the learning-engendering conflict differ in their respective implications regarding the way the conflict is to be resolved. The acquisitionist vision of conflict resolution is grounded in the principle of noncontradiction—in the assumption that any two narratives sounding as mutually contradictory are also mutually exclusive, and that there is a common criterion for deciding which of them should be rejected and which one endorsed and labeled as true. Commognitive conflict, in contrast, is defined as the phenomenon that occurs when seemingly conflicting narratives come from different discourses—from discourses that differ in their use of words, in the rules of substantiation, and so on. Such discourses are *incommensurable* rather than incompatible, that is, they do not share criteria for deciding whether a given narrative should be endorsed.[10] Unlike in the case of conflicting narratives coming from the same discourse, two narratives that originate in incommensurable discourses cannot automatically count as mutually exclusive even if they sound contradictory. This kind of conflict, therefore, cannot be resolved with a decisive empirical evidence, confirming one of the conflicting claims and refuting the other. Thus, whereas acquisitionists view conflict resolution as making sense of the world, commognitivists regard it as making sense of other people's thinking (and thus talking) about this

[10]By *commensurable*, says Rorty (1979), "I mean able to be brought under a set of rules that which will tell us what would settle the issue on every point where statements seem to conflict" (p. 316). In other words, incommensurability means there is no super-theory that provides criteria for proving one framework right while refuting the other. "Incommensurability entails irreducibility [of vocabularies], but not incompatibility" (p. 388).

TABLE 1
Comparison of Concepts: Commognitive Conflict Versus Cognitive Conflict

Concept	*Cognitive Conflict*	*Commognitive Conflict*
Ontology: The conflict is between	the interlocutor and the world	incommensurable discourses
Role in learning	is an optional way for removing misconceptions	is practically indispensable for metalevel learning
How is it resolved?	by student's rational effort	by student's acceptance and rationalization (individualization) of the discursive ways of an expert interlocutor

world.[11] This means a gradual acceptance, "customization," and rationalization—figuring out the inner logic—of other people's discourses.

The differences between the concepts of cognitive and commognitive conflict are summarized in Table 1.

Related Approaches

Whereas the commognitivist shares his or her interest in communication with many other investigators of learning, the way this researcher situates discourse among concepts such as thinking and learning is quite distinct. In the majority of current studies discourse is featured as a means for learning. It is this outlook that inspires such common statements as "Classroom discourse helps in learning mathematics." The commognitivist, in contrast, regards discourse as the very object of learning. This is a more radical view than what can be found in other places. Even those writers who made the decisive shift from the acquisitionist study of cognition to the participationist study of communication do not usually present the equation *thinking = self-communication* as explicitly as it was done on the preceding pages. Discursive psychology is probably the best known of those more radical approaches (see, e.g., Edwards, 1997; Edwards & Potter, 1992; Harré & Gillett, 1995). The main characteristic of this strand is its stress on the unity of thinking and speech (note that the commognitivist goes even further, in that he or she rejects

[11]Commognitive conflict is often involved also in mathematical invention (or any other scientific invention, for that matter). In this case, the conflict is likely to occur within a person, between two partially overlapping discourses in which the person is embedded. Indeed, in the transition from a familiar discourse to a new one, the mathematician may find him- or herself endorsing conflicting narratives. One well-known case of such inner conflict is that of George Cantor, the inventor of set theory, who in his letters to another mathematician, Richard Dedekind, complained about his inability to overcome the contradiction between the well-known "truth" that a part is smaller than the whole and the conclusion he had reached on the grounds of his new theory, according to which a subset of an infinite set may be "as big as" the whole set (Cavaillès, 1962).

not just the split between thinking and speech, but the more general one, between thinking and communicating). In this respect, commognition is also closely related to the contemporary semiotics, where argument is made not only against the thought–communication dichotomy, but also against the one between thought and content.[12]

When it comes to theories concerned specifically with mathematical learning and aimed at capturing the dynamics of this process in all its complexity, one of the most notable, best developed examples is the work on classroom norms done by Paul Cobb and his colleagues (see, e.g., Cobb, 1996; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Yackel & Cobb, 1996). Although Cobb's approach and the commognitive framework are close relatives—both are participationist and both have been created with the primary aim to tackle mathematical learning—the basic assumption that generates all the other commognitive tenets is unique to this latter framework. This foundational principle is bound to make a difference. True, Cobb and his colleagues' notions and assertions do have their commognitive counterparts. Thus, for example, the concept of *norm*, which is central to Cobb and his colleagues' work along with the distinction between *mathematical*, *social*, and *sociomathematical*, has been defined within the commognitive framework as a subcategory of metadiscursive rules.[13] Subsequently, Cobb's distinction between mathematical and sociomathematical norms was replaced by the commognitive distinction between object-level versus metalevel norms. These translations, however, underlie an isomorphism rather than a full-fledged equivalence of approaches. Indeed, if the meaning of a word lies in its relation to the rest of its "native" vocabulary, such translations cannot be regarded as merely "saying the same

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[12]This distinction seems unquestionable as long as the talk is about concrete material objects. And yet, as has been argued by many writers (Dreyfus & Rabinow, 1982; Foucault, 1972; Gottdiener, 1995; Sfard, 2000b), the divide between thought and its referent loses ground when it comes to more abstract objects, such as numbers (some writers go on to question even the ostensibly unproblematic case of the talk about concrete objects). One can argue that abstract objects are, in fact, metaphors inspired by the discourse on material reality: They come into being when we replace discursive processes with nouns and then use these nouns within phrases modeled after the discourse on concrete objects. Think, for example, about the use of the noun *[number] five* as a substitute for counting up to 5 or about the term *function* x^2, which we use when trying to say something general about the operation of squaring numbers. In this context, think also about the phrases such as, "Given a function…" and "There are numbers such that…"—expressions that can be read as implying an existence of extradiscursive entities for which the nouns are but linguistic pointers. As a result of a prolonged use of such objectifying discursive forms, the putative entities often become experientially real to the user, who starts act upon them as if they were a part of a mind-independent reality. Mathematical discourse turns out to be an *autopoietic* system—a system that produces its own objects.

[13]To be considered a norm, the rule must fulfill two conditions. First, it must be widely enacted within the community of the given discourse, such as a school class or the community of research mathematicians. Second, it must be endorsed by almost everybody, and especially by expert participants. If discussed by experts, this meta-rule must be explicitly presented as one of the defining properties of the given type of discourse.

thing in a different way." To give just one example, because discourse is an inherently social activity that produces its own objects, object-level rules of mathematics should not count as any less social than all the others. This is not, however, what seems to be implied, possibly inadvertently, by the terminology of mathematical versus sociomathematical. These differences notwithstanding, the commognitive research and the work done by Cobb and his colleagues are closely related and mutually inspiring.

Having made these conceptual preparations, I am now in the position to show the commognitive framework in action. Let me thus turn to the empirical studies. In the analyses that follow, I focus on only one type of the required discursive change: the change in certain crucially important metadiscursive rules. Because of its tacitness, this kind of change has not been explicitly investigated so far. The point I make throughout my account is that visible classroom occurrences may, in fact, be the tip of an iceberg, whereas in order to understand learning, its mechanisms, and its impediments, one needs to dive under the surface and examine those aspects of communication that usually go unnoticed.

FIRST STUDY: LEARNING ABOUT NEGATIVE NUMBERS[14]

Our wish to investigate students' first encounters with negative numbers was motivated by the belief that this topic was somehow unique among mathematical subjects learned at school. We felt that for many students, negative numbers were particularly challenging, and not necessarily because of the intricacy of arithmetical techniques involved. Our conjecture was reinforced by, among others, the autobiographical account of the French writer Stendhal,[15] who, in his memoirs, recalled his difficulty with understanding the source of the claim that "minus times minus is plus." "That this difficulty was not explained to me was bad enough," he said. "What was worse was that it was explained to me by means of reasons that were obviously unclear to those who employed them" (quoted in Hefendehl-Hebeker, 1991, p. 27). As, according to Stendhal himself, his teachers were certified "mathematical luminaries," the claim that they were unsure of their reasons did not sound convincing. We conjectured instead that the teachers' reasons were not Stendhal's own: To be persuaded, Stendhal needed a different kind of justification. Our classroom investigation was driven by, among others, the wish to shed light on this puzzling, seemingly unbridgeable disparity between teachers' offerings and students' needs. We believed that the importance of what we would learn while unraveling this quandary would go beyond the special case of negative numbers.

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[14]This study was conducted with Sharon Avgil, who also served as the teacher.

[15]Pseudonym of Marie-Henri Beyle (1783–1842).

The study took place in a typical Israeli junior secondary school in a middle-class area. The class of 12- to 13-year-olds was observed in the course of 30 1-hr meetings devoted to negative numbers. The teacher's expositions and whole-class discussions were videotaped and audiorecorded. In addition, during times when the students were working in small groups, a camera was directed at two designated pairs. These two pairs were also regularly interviewed before, during, and after completing the learning sequence. The interactions, which were all held in Hebrew, were transcribed in their entirety and, for the sake of this article, partially translated into English. Because the aim of the study was to observe learning rather than to assess instruction, the teacher was given a free hand in deciding about the manner in which to proceed. Her teaching turned out to be guided by the principle of always probing students' own thinking before presenting them with other people's ready-made ideas. This principle clearly manifests itself already in Episode 1_N, presented below, in which the teacher tried to elicit what knowledge about negative numbers the students might have already possessed at the time she started to teach the topic:

Episode 1_N: The first lesson on negative numbers

1	Teacher:	Have you heard about negative numbers? Like in temperatures, for instance?
2	Omri:	Minus!
3	Teacher:	What is minus?
4	Roi:	Below zero.
5	Teacher:	*Temperature* below zero?
6	Sophie:	Below zero … it can be minus 5, minus 7 … Any number.
7	Teacher:	Where else have you seen positive and negative numbers?
8	Omri:	In the bank.
9	Teacher:	And do you remember the subject "Altitude"? What is *sea level*?
10	Yaron:	Zero.
11	Teacher:	And above sea level? More than zero?
12	Yaron:	From 1 meter up.

This conversation shows that at the time the learning began, the term *negative number* was not entirely unknown to the children. Because, however, students' participation was limited to one-word responses to teacher questions—the phenomenon that was observed also in earlier interviews—it is reasonable to summarize the children's initial skills in the following way: The students could identify the discourse on negative numbers when they heard it; they could associate the notion with some other expressions, such as *minus* or *below zero*; they could even respond in a seemingly reasonable way to some questions involving negative numbers. This said, they were not yet likely to formulate full-fledged statements on negative numbers on their own.

For the next 3 months, we observed the students as they became increasingly proactive and linguistically accurate in conversations featuring such new terms as *minus 2* or *minus 3½*. We were particularly attentive to the question whether the children's use of these terms was *objectified* (the term is explained shortly). We also watched the learners operating on specially designed visual mediators—extended number line, arrow model of negative numbers, and magic cube model.[16] While doing so, we tried to discern the slowly evolving mediating routines. Here, our main question was whether the children used the different mediators interchangeably. Finally, we documented the growing repertoire of narratives endorsed by the students, as well as the transformations that occurred—or failed to occur—in the children's discursive meta-rules. Full results of our study were reported in Avgil (2004). In this article, I focus on the change that was most likely responsible for Stendhal's complaints, the one that has been long known as a major challenge to many students. The three questions formulated in the introduction to this article are now answered with regard to this special aspect of learning.

Focus on the Discourse (Mathematics): What Kind of Change in Endorsement Routines Was Expected to Occur as a Result of Learning?

Stendhal's story cued us toward probably the most significant change that was about to occur in the children's mathematical discourse. We conjectured that if Stendhal found the claim "minus times minus is plus" insurmountably challenging, it was probably because he could not figure out where this claim had come from and why it had been endorsed; and if the substantiation offered by the teachers did not help, it was probably because their argument was not of the type that young Stendhal would find convincing. We concluded that the required change was in the meta-rules of endorsement or, more specifically, in the rules according to which one was supposed to decide whether to accept a given mathematical definition. Let me elaborate.

One possible way in which one may substantiate the claim "minus times minus is plus" is presented in Figure 1. The argument originates in the principle that the extended discourse must preserve some critical features (object-level rules) of the original numerical discourse. Basic rules of addition and multiplication (associativity, commutativity, distributivity, etc.) were identified by mathematicians of the past as the ones that epitomized the nature of numbers. These were, therefore, the properties that were chosen to be retained. Subsequently, these predetermined object-level invariants were called *axioms* of the numerical field. In Figure 1, the rule

[16]An arrow model presents a signed number as a vector on the number line, the length of which equals the absolute value of the number and the direction of which corresponds to the sign (the vector corresponding to a negative number points left, the one corresponding to the positive points right). Magic cubes are entities that, when inserted into a liquid, increase (in case of the positive numbers) or decrease (for negatives) the temperature of this liquid by 1 degree.

Taking as a point of departure the request that the basic laws of numbers, as have been known so far, should not be violated, and assuming that the rules $a \times (-b) = -ab$ and $-(-a) = a$ have already been derived from these laws, we may now argue that for any two positive numbers a and b, the following must hold:

On the one hand,

$$0 = 0 \times (-b) = [a + (-a)](-b) \quad (1)$$

and on the other hand, because of the distributive law which is supposed to hold,

$$[a + (-a)](-b) = a(-b) + (-a)(-b). \quad (2)$$

Because it was already agreed that $a(-b) = -ab$, we get from (1) and (2)

$$-ab + (-a)(-b) = 0.$$

From here, and from the law $-(-x) = x$, one now gets

$$(-a)(-b) = -(-ab) = ab$$

FIGURE 1 Deriving the rule for "minus times minus is plus" from the basic rules (axioms) of numerical discourse.

"minus times minus is plus" is derived as a necessary implication of this requirement.

The speculation that the substantiation given to Stendhal by his teachers, whatever its actual form, followed a similar path is highly plausible simply because no other argument seems available. In particular, there is no concrete model from which this rule could be deduced.[17] If so, it is quite clear why Stendhal was hesitant to accept the explanations and, more generally, why other learners are likely to go through a similar experience: For those who have known only unsigned numbers so far, a concrete model has always been the stepping stone—and the ultimate reason—for mathematical claims. Indeed, before the appearance of negative numbers, mathematical and colloquial discourses were unified in their endorsement routines: In both cases, the narratives were verified by confronting propositions in question with discourse-independent reality. Consequently, decisions about the endorsability of mathematical statements were perceived by the participants of

[17]On the face of it, this claim may be contested, because many ideas have been proposed to model negative numbers (e.g., there is the model of movement whereby time, velocity, and distance can be measured in negative as well as positive numbers; numbers may be represented as vectors). And yet, upon closer look, all of these explanations turn out to be derivatives of the same basic decisions about preserving certain former rules of numbers while giving up some others; these fundamental choices are exactly the same as the ones that find their expression in the acceptance of axioms of numerical field as a basis for any further decision, and they must be accepted, possibly in a tacit way prior to any justification (see also Sfard, 2000a).

mathematical discourse as imposed by the world itself. This impression was fortified by the fact that the mathematical discourse was fully objectified, that is, mathematical nouns and symbols, such as *three*, *half*, or *2.35* were treated as objects in their own right, with all the traces of human agency removed from mathematicians' stories about them. The substantiation routine of the new discourse, instead of pointing to mind-independent, extradiscursive reasons for the endorsement of the claim "minus times minus is plus," rests on the exclusive attention to the inner coherence of the discourse. This is a rather dramatic change in the rules of a mathematical game—and a major challenge to the learners. The nature of the change, and thus of the commognitive conflict likely to spur learning, is presented schematically in Table 2.

The task will not be easy, if only because of the nonexplicit nature of the conflict. An additional difficulty stems from the fact that in the process of extending the numerical discourse, preserving some former discursive features goes hand in hand with compromising some others. Among the numerical properties that mathematicians agreed to give up in the transition to the signed numbers were those that involved inequalities. For example, in the extended set of numbers the claim "If $a > b$ then $a/b > 1$ for every a and $b \neq 0$" is no longer true. Mathematicians' tacit criteria for deciding what to preserve (and call *axiom*) and what to give up cannot possibly be clear to children. A cursory look at the history of negative numbers suffices to see that for a long time, these criteria were far from obvious to the mathematicians themselves. The fact that the negatives lacked some of the properties that, so far, had appeared as the defining characteristics of numbers led Chuquet, Stifel, and Cardan to claim that the negatives were "absurd," "false," "fictitious," and "mere symbols" (Kline, 1980, p. 115). Nearly two centuries later Descartes stated that these numbers were "false, because they represent numbers smaller than nothing," whereas Pascal declared, "I know people who don't understand that if we subtract 4 from zero, nothing will be left" (ibid). Back then in the 17th century the real, albeit unspoken, question was about the rules of mathematical game: Who is the one to decide what counts as mathematically acceptable—the reality itself or the participant of the mathematical discourse? Hundreds of years passed before this dilemma was finally resolved. Our study was to show how contemporary students and teachers make it—or fail to make it—through this complex developmental juncture.

TABLE 2
Old and New Mathematical Meta-Rules for Endorsement of Definitions

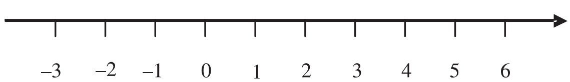
Old Meta-Rule	*New Meta-Rule*
The set of object-level rules to be fulfilled by the defined object must be satisfied by a concrete model.	The set of object-level rules to be fulfilled by the defined object must be consistent with a predetermined set of other object-level rules called axioms.

Focus on the Process: How Did the Students and the Teacher Work Toward This Change?

Teaching: Helping children out of the inherent circularity of discourse development. A certain inherent circularity of the discurse development was likely to obstruct students' learning from the very beginning. To illustrate, let us look at the introduction to the topic taken from a typical textbook (Figure 2). The crux of this definition is in the interesting conceptual twist: Points on the number line are marked with decimal numerals preceded by a dash, and, subsequently, these marked points are called *negative numbers*. One may wonder about the reasons for these verbal acrobatics: giving new names to *points* on a line and saying these are *numbers*. Whereas it is virtually impossible to introduce a new discourse without actually naming its objects from the very beginning, it is also very difficult to use the new names without anchoring them in something familiar. Alas, negative numbers are not anything that could be associated with easily identifiable referents. Unlike in other discourses, where one can indicate a new object by referring the students to some familiar perceptual experience (think, e.g., about teaching velocity or exotic animal species), in the discourse on negative numbers the initial remarks on the new mathematical objects have almost no concrete instantiations to build on. Points on the extended number line, although far from sufficient, is probably the best visual mediator one can think of in this very first phase of learning.

The process of introduction to the new discourse is thus inherently circular: Although the learning sequence that begins with giving a new name to an old thing seems somehow implausible, it can hardly be avoided simply because even the first step in a new discourse is, by definition, already the act of participation in this dis-

Let's choose a point on a straight line and name it "zero." Let's choose a segment and call it "the unit of length." Let's place the unit head-to-tail repeatedly on the line to the right of the point "zero." *The points made this way will be denoted by 1, 2, 3 and so on …*



To the left of the point "zero," we put the unit segment head-to-tail again and *denote the points obtained in this way with numbers –1, –2, –3, … The set of numbers created in this way is called the set of negative numbers.*

FIGURE 2 From a school textbook (Mashler, 1976). Translated from the Hebrew by Anna Sfard).

course. This, however, faces the learner with a dilemma: On the one hand, in order to objectify the new number words and see them as numbers, not just labels, the student needs to use these words "the numerical way" (i.e., has to speak about adding them, multiplying, etc.); on the other hand, how can a person talk "the numerical way" about something that is not yet seen as a number? The learner's dilemma becomes the teacher's challenge: The educator's task is to help the children out of the circularity. The teacher has to find a way to break the vicious circle and make the students actually talk about the negatives even if the young interlocutors do not yet have a full sense of the new entities' number-like nature.

In our study, the teacher's solution was to provide the students with additional tools to think with. She introduced familiar visual mediators about which the children would be able talk without much explanation and that would generate a discourse very similar to, perhaps even identical with, the talk on negative numbers. The choice of the mediators was made carefully, so as to ensure they would not be treatable in terms of the "old" (unsigned) numbers more easily than in terms of the new ones (as is the case with the majority of real-life situations supposedly supporting the use of negative numbers; e.g., questions about changes in temperatures do not, in fact, necessitate manipulating negative numbers). As mentioned previously, three such mediators (number line, arrows, and magic cubes) were introduced. The teacher's assumption was that the children, once provided with such self-explanatory generators of the relevant talk, would be able to make much progress on their own. She hoped that they would arrive at the generally accepted rules for adding and subtracting signed numbers and at the rule for multiplying a negative number by a positive. It was the "minus times minus" question that was supposed to be the site of the major commognitive conflict. After all, it is the only case that cannot be derived from a model. Classroom events, however, took an unexpected course.

Learning: Breaking out of the discursive circularity by recycling old routines. The models did help the students in making their first steps in the new discourse. For better or worse, the children seemed to know what to expect from something that has been labeled as *number*: When asked to perform a number-like operation on positive or negative arrows or cubes, they summoned discursive routines associated with the old numbers. This reliance on the former discursive habits could be seen in the following conversation between two students, Sophie and Adva, who at this point were already well acquainted with both mediators and knew how to add and subtract signed numbers. They were now trying to figure out how to multiply a positive by a negative:

Episode 2_N: The children try to find the value of (+2) × (–5)

1121	Sophie:	Plus 2 times minus 5 …
1122	Adva:	2 times minus 5

1123	Sophie:	Aha, hold on, hold on, plus 2 … it is as if you said minus 5 multiplied 2 times [*looks at the written expression: (+2) × (–5)*]. So, minus 5 two times it is minus 10 …
1124	Adva:	How about plus 2? How about the 2?
1125	Sophie:	[*looks at the written text*] Minus 5 … 1, 2, 3, 4, 5 [*counts notches on the number axis left to the zero and eventually marks the fifth of them with "–5"*] Times 2. You know that plus 2 is 2, you can take the plus away, right? So it is like 2 times minus 5, 2 times minus 5, so it is minus 5 and one more [add[18]] minus 5 [*turns to Adva*]. It gives minus 10.
1126	Adva:	I don't know … 2, the plus—maybe it does mean something.
1127	Sophie:	Ok, you can take this plus away.
1128	Adva:	So, it is like I can take this minus away.
1129	Sophie:	No, not the minus, because this means 2 times minus 5.

So far, so good. In concert with the teacher's prediction, the concrete model and expectations evoked by the word *number* helped the students find their way into the new discourse. Although not without some telling hesitation, Sophie and Adva were able to arrive at the generally accepted, historically established formula. They did it by projecting in a metaphorical manner from their former discursive experience into the new, unfamiliar context: In the realm of unsigned numbers, multiplication of a number by 2 meant adding that number to itself, and they used the same interpretation now, when the doubled number was negative. However, during the whole-class discussion that followed the work in pairs, not everybody shared their opinion:

Episode 3_N: In response to the question "What could (+2) × (–5) be equal to?"

1226	Roi:	Minus 10.
1227	Teacher:	Why?
1228	Roi:	We simply did … 2 times minus 5 is minus 10 because 5 is the bigger number, and thus … uhmm … It's like 2 times 5 is 10, but [it's] minus 10 because it's minus 5.
…	…	…
1248	Noah:	And if it was the positive 7 instead of positive 2?
1249	Yoash:	Then it would be positive 35.
1250	Sophie:	Why?
1251	Yoash:	Because the plus [the positive] is bigger.

At the first sight, Roi's idea might sound surprising. Upon closer look, it was grounded in the principle of preserving the rules of the former discourse, similar to the one that had guided Sophie. As presented schematically in Figure 3,

[18]The Hebrew expression *ve-od*, which literally means "and [one] more," is used in school mathematics in the sense of "add" or "plus" (the word *plus* itself may also be used).

Successful try: substitution into the discursive template

$2 \times b = b + b$

Unsuccessful try: substitution into the discursive template

$$(+a) + (-b) = \begin{cases} |a-b| & \text{if } a > b \\ -\ |a-b| & \text{if } a \leq b \end{cases}$$

in which a and b are unsigned and both + and – are substituted with ×.

FIGURE 3 Recycling old discursive templates in the new context.

Sophie substituted the new numbers for the old numbers: In the familiar multiplication procedure for unsigned numbers, the negatives had slid into the slot of the second multiplier, occupied so far exclusively by unsigned numbers. In Roi's case, the new task evoked the formerly developed routine for the addition of signed numbers and the students substituted operation for operation: The multiplication of signed numbers was obtained from the multiplication of unsigned numbers in the way in which the addition of signed numbers had been obtained from the subtraction of the unsigned, more or less. To sum up, Roi, just like Sophie before him, drew on previously developed discursive routines, except that his choice did not fit with the historical decision made by the mathematical community.

In an attempt to account for the difference in Sophie's and Roi's choices of the rule to preserve, let us take a closer look at the two children's discourse on negatives. Roi's ways of talking were not unlike those of Adva in Episode 2_N or of Yoash in Episode 3_N. Nothing indicated that any of these children had objectified negatives, that is, could speak about them the way they spoke about more familiar numbers, as self-sustained entities remaining in a numerical relation one to another and to the other numbers. On the contrary, evidence abounded that the signs + and – had not yet turned for any of them into an integral part of the names of number-like entities. In Utterances 1126 through 1129, the children discussed taking the sign "away." This, by itself, might not be sufficient evidence for the lack of objectification. And yet, the question "Maybe [the plus] does mean something?" (1126) asked by Adva when she tried to decide whether to delete the sign from +2 showed that, for her, only 2 deserved being called *number*, whereas the sign was somehow tacked on and not necessarily relevant for the course of numerical conversation. In other places, children who shared Roi's idea about multiplication could be heard using phrases such as "the plus [the positive] is bigger" (1251) with respect to the pair +7 and –5. Judging from the context in which the announcement about the "bigger" number was made, it resulted from the comparison between "numbers without the signs." All this indicated that in expressions such as +7 or –5, only the numeral part counted as a number, whereas the sign was something that did not affect this numerical identity anymore than, say, the change in a name or in the external appearance affects one's identity as a person.[19]

Similar analysis of the way Sophie used numerical words shows that in contrast to her classmates, the girl had already made a significant step toward objectification of the talk on negatives. This was particularly salient in the episode that follows, taken from the whole-class discussion:

Episode 4_N: Sophie's response to the question "What could 6 × (–2) be equal to?"

1364	Sophie:	I say that if you have one minus and one plus, then you go with the plus, that is, if you have here minus 2 times plus 6 [*(–2)* × *(+6)*], then you do 6 times minus 2 … [*6* × *(–2)*]
1365	Teacher:	You mean, I need to reverse their order?
1366	Sophie:	The order here doesn't matter.

This brief conversation brings into an even stronger relief what could be seen already in Episode 2_N: Sophie could treat expressions such as "negative 5" (or "minus 5") as integrated wholes and was capable of incorporating them into the numerical discourse simply by putting them into slots reserved for numbers (see Figure 3). Moreover, the fact that she had little difficulty extending the endorsed narrative "*2 times a number* means adding this latter number to itself" to the negatives adds plausibility to the conjecture that for her, these new entities were as "add-able" as the numbers she knew before. This claim finds its further reinforcement in Sophie's naturally adopted assumption about the commutativity of the extended multiplication (see her Utterance 1357: "Order doesn't matter"). In contrast, Roi, for whom –2 and –7 did not yet represent integrated entities that deserved being called *numbers*, was more arbitrary in his projections from the old numerical discourse to the new one. The rule he chose, according to which the numeral part of the numerical symbol and the sign attached to the numeral were to be treated separately, reflected his "split vision" of the negatives as "numbers with signs attached."

As a result, the class was certainly facing a commognitive conflict, albeit not the one that we identified earlier as the crucial and inevitable hurdle on their way toward the discourse on negative numbers. Indeed, neither Sophie nor Roi were enacting the new routine, thus the new meta-rule for defining, as presented in Table 2. Sophie still relied on the old principle of referring to the evidence of concrete

[19]This phenomenon is certainly reinforced by the fact that positive numbers are usually presented without the sign, e.g., as 2 rather than +2, and subsequently, a symbol such -2 is interpreted as the (positive) number 2 with the minus attached. Also note that as long as the term number was not objectified, children use this term with reference to written symbols (strings of digits) rather than to "intangible entities' supposedly "represented" by these symbols.

model, whereas Roi, who seemed to have renounced that meta-rule, did not regulate his decisions with any predetermined set of axioms chosen according to a justifiable set of criteria.

Teaching: Transition to the "telling" mode. Whether desired or not, there was a conflict, and it needed to be resolved. The students had now to decide which of the resulting incompatible narratives should be endorsed and which had to be disqualified as inappropriate. However, the lesson ended soon after the introduction of the two proposals and before the class had an opportunity to reach a resolution. The teacher, being convinced that the children would soon find out their way out of the momentary confusion, regretted to have lost the opportunity to watch the process in its entirety. Later that day she wrote in her journal:

> The lesson ended and I had to let the children go. I am afraid that they will check it at home and I will lose the opportunity to listen to their further thinking. But I have no choice. I don't give them any homework and hope to resume our conversation in two days, exactly from the point where it ended today.

The teacher's fears did not materialize, though. The next lesson began with the whole-class debate, and it was clear that the disagreement about the "plus times minus" persisted. The teacher hoped, however, that the explicit confrontation between the two alternatives would soon lead the class to the unequivocal decision about the preferability of Sophie's proposal. The following excerpt from this conversation aptly instantiates the general spirit of the lengthy debate that followed:

Episode 5_N: Trying to decide between the two proposals for "plus times minus"

1341	Teacher:	Come on, let's take the expression … minus 2 times 6 [*writes* "(–2) × (+6) =" *alongside the expression* "(+2) × (–5) =" *already on the blackboard*]. What is the answer and why?
1342	Naor:	Plus 12, because 6 is bigger than 2.
1343	Teacher:	Plus 12 [*writes on the blackboard* "(–2) × (+6) = 12?"]. I added the question mark because we don't know yet.
1344	Student:	When will you tell us?
1345	Teacher:	I will tell you today, but … in fact, what is your opinion? What do you say, Vladis?
1346	Vladis:	Me too: Plus 12, because 6 is bigger.
1347	Teacher:	What do you say, Sophie?
1348	Sophie:	I say *minus* 12.
1349	Teacher:	[*writes* "(–2) × (+6) = –12"] Why?
1350	Sophie:	Because you can take the plus of the 6 away and then you get 6 times minus 2.

1351	Roi:	But you can do the opposite.
1352	Teacher:	What do you mean by “the opposite”?
1353	Roi:	You can do 2 times plus 6. Why do we have to do 6 times minus 2?
1354	Teacher:	Because 2 has the minus.
1355	Roi:	So what?
1356	Teacher:	Are you saying that I should ignore also these parentheses? [*points to the parentheses around* –2]
1357	Sophie:	What does it mean “minus 2 times”? This is what you are saying [*she addresses Roi*]. You ignore the minus …
1358	Roi:	Ok, you have to make both of them plus or both of them minus.
1359	Teacher:	Do we have to “make them both plus or minus” …
1360	Roi:	Yes, somehow.
1361	Teacher:	… or should we decide whether the result is plus or minus?
1362	Roi:	In these two exercises [*points to the two expressions on the blackboard:* “(–2) × (+6) =” *and* “(+2) × (–5) =”] we decide according to the bigger.

This exchange is remarkable for at least two reasons. First, it shows that contrary to the teacher’s expectations, the children did not converge on Sophie’s proposal. Surprisingly, it was Roi’s version of the multiplication law that was winning the broader following. The second thought-provoking fact is the teacher’s restraint and her persistent refusal to step in with decisive judgment.

The classroom debate went on for another full period, and an even greater majority of students decided to give support to Roi, who continued to claim that the sign of the product should be like that of the multiplier with the bigger absolute value. Recurring demonstrations with arrows and magic cubes did not help. At a certain point, some of the children began showing signs of impatience: They were asking for the teacher’s authoritative intervention (1344). The teacher could no longer persist in her refusal to act as an arbitrator. Although initially reluctant (1345), she finally stepped in with the explicit ruling:

Episode 6_N: The teacher tells the children how to multiply numbers with different signs

[1372]	Teacher:	I want to explain what Sophie said. What she said is true, and this is the rule that guides us. Sophie did not manage to convince all of you, but I believe that some of you did get convinced that to multiply is to add time and again … for example, here [*points to* “(–2) × 6 =” *written on the blackboard*] you add the number –2 six times [*she marks arcs that symbolize –2 on the number line, from point 0 to the left*] … and I reach –12, and this is the right answer.

As if against herself, the teacher resolved the problem by revealing her own expert vision of who was right. True, there was an attempt at substantiating this decision by pointing to the repeated addition procedure. And yet, there was also something defeatist in the way the explanation was presented. The very fact that the teacher repeated the argument that had already been tried and had not worked for the children made her sound resigned and unconvincing. The teacher's disappointment found its expression in the note she made for herself after the lesson:

> In the beginning of the lesson I said to myself: "Fortunately, the children were not too interested in the topic … They are back without the answer…." [Now, after the lesson] I can see that even my repeated emphasis on the correct proposal did not help—the only thing that counts is the kids' wish to be like the leaders of the class.

The disillusionment as to the prospects of children's independent reconstruction of the numerical laws led to a change in the teacher's strategy. The last formula the class had yet to learn was that minus times minus is plus. Although the teacher did not give up the idea of letting the students probe their own creative ideas before being exposed to other people's discursive constructs, she was afraid of a lengthy discussion. This time the children were not expected to be able to reinvent the rule by themselves, anyway.

The students found the task of figuring out the product of –2 and –3 quite confusing. As can be seen in Episode 7_N, this was true even of Sophie, one of the few children who had little difficulty reinventing mathematicians' ways of multiplying positive numbers by negative:

Episode 7_N: Sophie and Adva try to figure out what (–3) × (–2) might be

1400	Sophie:	[*reads from the worksheet*] "How, in your opinion, can we perform each of the following operations and why …" And this is exactly what she said, minus 3 times minus 2 … Ok.
1401	Adva:	Minus 6, because they are both minus.… No, I don't understand … don't know what we are supposed to do.
1402	Sophie:	2 minus, see, do you remember how we did plus 4 times minus 2? You can delete the plus, [so we have] 4 times minus 2, you do 4 times minus 2, this is minus 8 … but now she gave us this worksheet so that we do operations with both [numbers] with minus. So, what do we do when both are minus?
1403	Adva:	Aah […]
1404	Sophie:	You can do minus 3 times minus 2, but what is "minus 3 times?"
1405	Adva:	3 times minus 2.
1406	Sophie:	But you have to consider the minus!
1407	Adva:	In this case there will be minus in the end.
1408	Sophie:	What? Do you think that you can erase the minus and do 3 times minus 2?
1409	Adva:	But this will be minus in the end in any case.…

1410	Sophie:	But I am not so sure about it. Look, you can perhaps do something like that: You can delete the minus [*points to the minus of the number –3*] and you get 3 times minus 2, and this gives minus 6—you think you can do this?
1411	Adva:	I don't know.
1412	Sophie:	I am not sure about this. Can you delete the minus when both are minus? This would mean that the result would be minus, and that you can erase the minus of the first or of the second …

This conversation did not seem to lead to anywhere. The girls were grappling in the dark, never sure of what they were saying. The teacher, anxious to spare her students additional frustration (or perhaps afraid that, as before, some of the children would develop an attachment to unwanted formulas!), decided to present her own answer. Always respectful toward students but unable to advance their own thinking any further, she opted for the second best: Rather than parachuting the new law on the class, she derived this rule from what the children already knew:

Episode 8_N: The teacher's intradiscursive substantiation of the laws of multiplication

1556a	Teacher:	Well, I wish to explain this [$2 \times (-3) = -6$] now in a different way.	[*writes on the blackboard the following column of equalities:* $2 \times 3 = 6$ $2 \times 2 = 4$ $2 \times 1 = 2$ $2 \times 0 = 0$ $2 \times (-1) = -2$ $2 \times (-2) = -4$ $2 \times (-3) = -6$ *While writing, she stops at each line and asks the children about the result before actually writing it down and stressing that the decrease of 1 in the multiplied number decreases the result by 2.*]
1556b	Teacher:	Let us now compute (–2) (–3) = in a similar way.	[*as before, writes on the blackboard the following column of equalities, stopping at each line and asking the children about the result before actually writing it down and noting that the decrease of 1 in the multiplied number increases the result by 3; this rule, she says, must be preserved when the left multiplier becomes negative:*

$$\begin{aligned} &3 \times (-3) = -9 \\ &\quad \mathit{2 \times (-3) = -6} \\ &\quad \mathit{1 \times (-3) = -3} \\ &\quad \mathit{0 \times (-3) = 0} \\ &\quad \mathit{(-1) \times (-3) = 3} \\ &\quad \mathit{(-2) \times (-3) = 6}] \end{aligned}$$ [20]

Summing up, one may say that in spite of the gradual evolution of the teacher's instructional strategies, two salient features of her way of teaching endured. First, she was deeply convinced that the students should play an active role in the advancing of mathematical discourse. This principle remained in force even when she had to compromise her initial intention to build on children's own inventions. In this latter case, while presenting to the students other people's discursive constructions, she did her best to make sure that nobody accepted what she was saying merely because of her privileged position as a teacher. Second, at no point did she attend directly to the metadiscursive rules for endorsement of narratives that influenced her decisions from behind the scenes and that, unnoticed, underwent a substantial change in the span of a few lessons. These rules were left hidden even when the law of multiplying two negatives was discussed. As a result the children, although exposed, this time, to the proper type of metalevel conflict, remained unaware not only of its nature, but even of its very existence.

Focus on Effects: Has the Expected Change Occurred?

A number of questions have to be asked now. How effective was the teacher's attempt to introduce the new endorsement routines simply by implementing them? Can the children satisfy themselves with the inner consistency of mathematical discourse as the sole criterion for the endorsement of narratives about numbers? Students' reactions to the teacher's derivation of the product of two negatives demonstrate that this is not the case:

Episode 9_N: Children's reactions to teacher's derivation of the laws of multiplication

1557	Shai:	I don't understand why we need all this mess. Is there no simpler rule?

[20]Note that the teacher's argument is a school version of the one that has been presented formally in Figure 1: In both cases, the laws of multiplying signed numbers are derived from the laws that hold for unsigned numbers according to the principle of preserving certain basic features of numerical operations.

1559	Sophie:	And if they ask you, for example, how much is (–25) × (–3), will you start from zero, do 0 × (–3), and then keep going till you reach (–25) × (–3)?

Evidently, the children did not even recognize the purpose of the teacher's argument. Rather than viewing it as an attempt at mathematical substantiation, they interpreted the exposition as a demonstration of the routine for producing endorsed narratives such as (–2) × (–3) = 6 or (–25) × (–3) = 75, and a very cumbersome one at that. Unable to tell substantiation of narratives from their production, the students still had a long way to go until their endorsement routines underwent the necessary transformation.

This conclusion was reinforced by certain utterances made by the children in response to the teacher's recurrent queries about their reasons for choosing Roi's rule for "plus times minus." Here is a representative sample from the conversation that followed one such query:

Episode 10_N: The teacher tries to understand why the children opted for Roi's formula

1334	Teacher:	You repeat time and again what Roi said last time. I need to understand *why* you think this is how things work?
1335	Yoash:	Because this is what Roi said.
1336	Teacher:	But Roi did not explain why it is so—why it is according to the bigger …
1337	Roi:	Because there must be a law, one rule or another.
1338	Teacher:	Ok, there must be some rule. Does it mean that we should do it according to the magnitude?
1339	Leah:	Yeah … The bigger is the one that decides.

Roi's exclamation "there must be a law, one rule or another" (1337) shows that the children fully accepted at least one basic rule of the numerical discourse: They agreed that whenever one dealt with entities called *numbers*, there had to be formulas that would tell one what to do. However, the conversation makes it equally clear that the students had not yet developed routines for producing and substantiating such formulas. When faced with the request to look for laws of multiplication on their own, the students grappled in the dark. Leah's appeal to the universal rules of the world ("The bigger is the one that decides," 1339) is a reminder that the student's previous experience with numbers made her think about them the way in which she thought about concrete objects: as entities that existed in the world and were subject to extradiscursive laws of nature (and for many people, the latter type of laws includes rules that govern human societies).

A different message, the message about the role of human factors in shaping mathematical discourse, was conveyed by one child's remark about "those who invented mathematics" (1462):

Episode 11_N: Vladis's proposal for the value of (–2) × (–3)

1458	Vladis:	It is plus 6.
1459	Teacher:	Why?
1460	Vladis:	According to the rule.
1461	Teacher:	According to which rule?
1462	Vladis:	According to the rule of those who invented mathematics.

Later we found out that Vladis learned the new rules of multiplication from his mathematically versed father. For all we know, this was also how he came across the claim about these laws' human origins. Although probably far from truly convinced that mathematics was a human invention, some of the children admitted that social considerations played a role in their decision making. Suffice to recall Yoash's frank assertion that his preference for the unconventional formula for multiplication was motivated by its having been proposed by his friend Roi (see 1335 in Episode 10_N). Roi's own explanation for the fact that the class voted for his proposal was further evidence of the children's awareness of their sensitivity to social factors:

Episode 12_N: Why choose one template rather than another?

1374	Teacher:	6 times minus 2 is minus 12—is this too complicated?
1375	Roi:	But I am more charismatic … I managed to influence them all.

Roi's ability to influence his friends on the force of his charisma rather than rational argument does not necessarily indicate students' unawareness of what kind of argument counts as legitimate in a mathematics classroom. Rather, it may be taken as yet further evidence of their confusion and helplessness. However we interpret one classroom utterance or another, it is clear that the tacit upheaval in the rules of the mathematical discourse bewildered the children. Moreover, meta-questions asked by the teacher ("Why should the numbers be multiplied this way?" and "Why did you choose this rule?") were hardly the children's questions, ones that they would be likely to tackle on their own. So far, the students had not had to bother themselves with meta-quandaries to be successful with numbers. Once such questions were asked, the children lost confidence. Unsure of the rules of the game anymore, they were now prepared to follow the lead of anybody who appeared to have a sense of direction and could show a measure of self-assurance.

Summary of the Commognitive Interpretation of the Process of Learning About Negative Numbers

According to commognitive analysis, learning about negative numbers involves a transition to a new, incommensurable discourse. One of the main changes that must happen in this transition is in the meta-rules of endorsement (and more specifically, of defining). In the course of learning, the participants of our study were faced with two types of commognitive conflict, only one of which was of the expected type, and neither of which seemed to have turned into a true opportunity for learning. The first conflict arose between Sophie and Roi when they proposed differing schemes for multiplying positive and negative numbers. This conflict, however, was unlikely to lead to the required discursive change simply because none of the children were enacting the new rule of endorsement that comes with the discourse on negative numbers. Sophie utilized the old principle of relying on the evidence of concrete models, whereas Roi, although evidently not restrained by this principle anymore, made his choice of multiplication law without referring to any predesigned set of criteria (axioms) and, in particular, without bothering about the properties of commutativity, associativity, distributivity, and so forth. The second conflict, the one that occurred when the teacher was introducing and substantiating the "minus times minus" scheme, was the expected type of conflict, but it remained unrecognized. Indeed, the new meta-rule for endorsement was enacted by the teacher but not made explicit. As a result, the students were unaware of the metalevel change and looked for the sources of their bewilderment elsewhere.

In view of this, it is probably not surprising that by the end of the 3-month learning–teaching process students' rules of endorsement had not yet changed in the required way and the children remained confused about the status of negative numbers and of the mathematical operations on these numbers.

SECOND STUDY: LEARNING ABOUT GEOMETRIC FIGURES[21]

In this study, pairs of first graders were asked to identify figures such as triangles or rectangles. The original aim of the project was to collect snapshots of students' geometrical discourse. The interviews, however, soon turned interventional, with

[21]The study was conducted with Orit Admoni. As in the research on negative numbers, the interviews were in Hebrew. For the sake of this article, portions of the transcripts were translated into English.

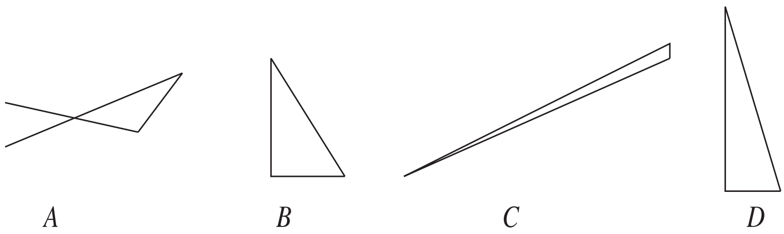


FIGURE 4 Which of these shapes is a triangle?

the interviewer acting as a teacher rather than a mere observer.[22] This unexpected transformation allowed us to observe yet another process of learning.

Focus on Discourse (Mathematics): What Kind of Change Was Supposed to Occur in Identification Routines as a Result of Learning?

In the Episode 1_T below, two 6-year-old girls, Shira and Eynat, dealt with basic geometrical concepts. The figures presented in Figure 4 were among the numerous geometric shapes presented on the page with which the children were working. The girls were asked to mark those items that could be called triangles. Once they completed the task, the following conversation took place:

Episode 1_T: The first meeting about triangles

[42:05]	Eynat:	[*pointing to Shape A*] This is a triangle but it also has other lines.
[42:08]	Teacher:	Well, Eynat, how do you know that triangle is indeed a triangle?
[42:12]	Eynat:	Because it has three … aah … three … well … lines.
…		…
[44:00]	Teacher:	[*pointing to Shape B*] This one also: one, two, three …
[44:02]	The girls:	Yes.
[44:03]	Teacher:	So, *is* it a triangle? Why didn't you mark it in the beginning?
[44:05]	Eynat:	'Cause then … I did not exactly see it.… I wasn't sure [*while saying this, Eynat starts putting a circle also around Shape C*]
…		…
[44:10]	Shira:	[*looking at Shape C that Eynat is marking*] Hey, this is not a triangle. Triangle is wide and this one is thin.
[44:16]	Eynat:	So what? [*but while saying this, she stops drawing the circle*]
[44:30]	Teacher:	Why? Why isn't this a triangle? [*points to Shape C*] Shira said it is too thin. But haven't we said …

[22]The use of the word *teacher* instead of *interviewer* in the description of this study is also useful—this linguistic move will allow me later to make some generalizing statements about teaching on the basis of the two studies presented in this article.

[44:35]	Eynat:	There is no such thing as too thin. [*but while saying this, she erases the circle around Shape C*]
[44:37]	Teacher:	Triangle—must it be of a certain size?
[44:41]	Shira:	Hmmmm … Yes, a little bit … It must be wide. What's that [*points to C*]? This is not like a triangle—this is a stick!

Here, unlike in the case of negative numbers, the students were already well acquainted with the mathematical objects in question, the triangles. And yet, neither the way they spoke about these shapes nor the manner in which they acted with them were fully satisfactory from the teacher's point of view. In her search for triangles, Shira disqualified any shape that seemed to her too thin. Eynat, even though apparently convinced that "there is no such thing as too thin" (44:35), still could not decide whether the stick-like shape in the picture was a triangle.

None of this comes as surprise to those who are familiar with the seminal work of Pierre and Dina van Hiele (van Hiele, 1985, 2004). In van Hiele's language, the resistance to the idea that the elongated shape may be called *triangle* shows that the children were still at the level of analysis: Although perfectly capable of "taking a figure apart" into a set of distinct elements, they were not yet able to distinguish between necessary and sufficient conditions for a figure to be a member of a given category. The ability to formulate and use definitions in the activity of identifying is a hallmark of the next level in the development of geometrical thought, known as the level of *abstraction* or of *informal deduction*. We accepted this description as one that can be easily translated into commognitive terms: van Hiele's levels may be interpreted as a hierarchy of mutually incommensurable geometric discourses, differing in their use of words and mediators, in their routines, and in the narratives their participants are able to construct and endorse.[23] This translation allows us to benefit from van Hiele's unique insights while disposing of the acquisitionist undertones of this theory and greatly increasing the resolution of the portrayal through detailed discourse analysis. This latter analysis allows an access to hidden obstacles that obstruct learners' progress.

In our present study, the main focus was on the characteristic that, according to our interpretation, was to undergo the most dramatic and most difficult transformation: As in the case of negative numbers, we concentrated out attention on a certain change in the metadiscursive rules that had to happen in the transition from van Hiele's level of analysis to the level of abstraction. Indeed, although in Episode 1_T the teacher may have appeared as having in mind not much more than a marginal

[23]van Hiele (2004) seemed only a few steps away from such commognitive rendering of this theory when stressing that the learners "speak a very different language" than the teachers (p. 62), that "each level has its own symbols and relations connecting these signs," that "[a] relation which is 'correct' at one level can reveal itself to be incorrect at another," and that "[t]wo people who reason at two different levels cannot understand each other" (p. 63).

change in the girls' word use, a closer analysis reveals that her effort could not be effective unless the change extended to the metadiscursive levels. As it turns out, being able to admit that a stick can also be called a triangle is a matter of a whole new type of discursive game. To have a better grasp of the imminent change, let us look at a new episode, the one in which the long debate on the status of the stick-like shape reached its climax:

Episode 2_T: Trying to convince Shira that Shape C is a triangle

[44:44]	Teacher:	But you told me, and Shira agreed, that in triangle there must be three lines, right?
[44:53]	Eynat:	Right.
[44:54]	Teacher:	So, come on, tell me how many lines do we have here? [*points to C*]
[44:56]	Shira:	One, two, three …
[45:09]	Teacher:	So, maybe this *is* a triangle? Here you said this one *is* a triangle [*shows another, more "canonic" triangle*].
[45:17]	Shira:	Because this one is wide and is like a triangle. It is not thin like a stick [*illustrates with hand movements and laughs*]
[45:24]	Teacher:	How do we know that a triangle … whether a shape is triangle? What did we say? What do we need in order to say that shape is a triangle?
[45:28]	Shira:	Three points … three vertices … and …
[45:32]	Teacher:	Three vertices and … ?
[45:37]	Shira:	Three sides.
[45:41]	Teacher:	And three sides. Good. If so, this triangle [*points to C*] fits. Look, one side … and here I have one long side, and here I have another long side. So, we have a triangle here.
[45:52]	Shira:	And … one vertex, and a second vertex, and a … point?!
[45:57]	Teacher:	Look here: one vertex, second vertex, third vertex.
[46:01]	Shira:	So it *is* a triangle?

I suspend for now any comment on perhaps the most striking feature of this exchange, namely the child's resistance to the idea that the long thin shape could be a triangle, and ponder instead the difference between how the children identified figures and how they should have been doing this according to the teacher. The act of *identification* can be divided, at least in principle, into two more basic components: the act of *recognition* that involves a recall of certain past experiences associated with the present one, and the act of *naming*—of attaching a word to the recognized shape. For Eynat and Shira, however, these two ingredients seemed to constitute an indivisible whole. Upon seeing certain two-dimensional figures, the girls uttered the word *triangle* spontaneously, without any former reflection. They identified the shapes the way they identified people's faces, that is, in an intuitive, nonlinguistic way and without being able to give reasons for their choices even to themselves. In the eyes of the teacher, this unpremeditated identification was no longer a satisfactory procedure. Within school mathematical discourse, children are requested to communicate to others not only their decisions but also the way these decisions are made. It is the matter of accountability that is the warrant of effective communication.

From now on, the identification procedures will thus have to be mediated by, and documented in, language. When the child tries to decide whether a polygon is a triangle, he or she will have to count its sides. The direct identification routine will have to give way to the composite one: First primary recognition, and then discursively mediated assignment, or just verification, of a name. The students will have to learn to suspend their spontaneous discursive decisions for the sake of reflective, metadiscursively mediated choices of words.[24] As our example seems to show, this is a difficult thing to learn. And no wonder. Although if judged by their results, the direct and the discursively mediated identifications would often remain indistinguishable, these two routines differ not only in their procedural aspects, but also in their ontological message and in their objectives. When as a result of the direct identification one says, “This is a triangle,” his or her aim is to state a truth about the world: One asserts that the shape he or she is talking about belongs to the “natural kind” of triangles. In this rendition, a shape is a triangle by the law of nature, not because of what we say. When the identification is discursively mediated, the utterance “This is a triangle” becomes equivalent to the metadiscursive sentence “This shape may *be called* a triangle.” In this latter case, we are making a statement about our own decisions rather than about the state of affairs in the world.[25]

This basic difference has many far-reaching consequences, but above all, it makes us feel differently about our rights and obligations with respect to the world of things under consideration. As long as we engage in the direct identification, our hands are tied. Unreflective choices of words come together with the deep conviction that our uses of the words are extradiscursively imposed. This is, no doubt, how Shira and Eynat felt while asked to identify triangles. Like in the anecdote about a child who, after learning about telescopes, was able to understand how astronomers discover new stars but still wondered about how they discover these stars’ names, the two girls seemed to regard geometrical shapes and the words that denote them as, in a sense, the same. No human decision could change for them what was or was not a triangle. But this was exactly the change that the teacher

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[24]These observations bring to mind Vygotsky’s (1987) famous distinction between spontaneous and scientific concepts.

[25]If the fact that definitions (e.g., “A triangle is a polygon with three sides”) and identifications (sentences of the form “This is a triangle”) are, in fact, meta-discursive statements (consider the fuller versions of the above two statements: “A polygon will be called a triangle if it has three sides” and “The name *triangle* can be applied to this shape”) often escapes us, it is because of the phenomenon that can be described as *ontological collapse*: our tendency to present meta-statements as if they were object-level assertions (compare the first pair of statements above with their “fuller” versions).

wanted to take place. What so far had appeared to the children to be a narrative determined by the word itself, the teacher was now trying to turn into a discourse about discourse. But the change was slow to come. In spite of the teacher's repetitive attempts, the ontological message of the statement "It is a triangle" was still quite different for the children from what it was for her.

The new discursively mediated routine for identification entailed a change of yet another metadiscursive principle. So far, assigning words to things had been an act of splitting the world into disjoint sets of objects. After all, as long as signifiers could not be separated from their signifieds, words had the property of the things themselves. This meant that no two words could be applied simultaneously to the same drawing, because no two material objects could occupy exactly the same space at the same time. Therefore, pointing to a shape and saying that this was both triangle and stick, or both square and rectangle, seemed out of question. All this would have to change with the transition to discursively mediated identification. The discursive procedures for mediated identification would now be ordered according to the relation of inclusion. For example, the procedure for identifying a square would be presented as including a procedure for identifying a rectangle: It would begin with counting the sides of the polygon and checking its angles, which is sufficient to find out whether the polygon is a rectangle; and it would continue with comparing the lengths of the sides, an action that is unique to squares. The hierarchical organization of these discursive procedures would become, in turn, a basis for the hierarchical categorization of geometrical figures. This and the formerly mentioned differences between direct and mediated identification are schematically summarized in Table 3.

TABLE 3
Comparison of Direct and Mediated Identifications

Variable	*Direct Identification*	*Mediated Identification*
Goal	To identify the "natural kind" to which the object belongs (to locate the object in the world)	To call an object an appropriate name
Procedure	Noncommunicable to others	Communicable (discursively mediated)
Level of narrative	Object level ("*x* is *y*")	Meta level ("*x* can be called *y*")
Ontological status of the act	Mind independent (the world imposes the use)	Human decision
Recognition vs. naming	The two combine into an inseparable whole	The two are separate acts
Relation to other names	Its extension does not overlap with the extensions of other words signifying geometrical shapes	Its extension occupies a well-defined place in the hierarchy of extensions

Focus on the Process: What Did the Participants Do to Make the Change Happen?

In the face of the girls' unsatisfactory use of the word *triangle*, the teacher did what in this situation seemed the most natural thing to do: She tried to make the incongruence explicit by summoning the definition of triangle:

Episode 3_T: Summoning the definition of triangle

[42:35]	Teacher:	Only four triangles. Ok, Eynat, how do you know that triangle is, indeed, a triangle?
[42:40]	Eynat:	Because it has three … well, lines.
[42:43]	Teacher:	It has three lines?
[42:45]	Eynat:	Yes.

As is evident from this brief exchange, there was no problem with the definition. The girls knew what it said and were able to check whether its requirements were fulfilled. And yet, it soon became clear that the definition did not do the job it was expected to do. In spite of the apparent consensus about the requirements that determine the figure's name, Shira was not yet ready to agree to what by now should have been obvious, at least according to the teacher: that the elongated figure was a triangle. No wonder. Before this transition might happen, the girls could not be aware of the decisive role of definition. As long as the children yielded to the spontaneous, unmediated identification, the condition furnished by the definition was considered to be necessary, but not sufficient.

The teacher, however, did not seem to realize the nature of the problem and the enormity of the task she was facing. This left her with little options. In fact, all she could think of was exemplifying her own routine for identification over and over again, asking the girls to perform the procedure with her. Each time, after counting polygon elements and uttering the words "one, two, three," she paused, added the telling "So …," and eventually, after another brief pause, asserted that the shape was a triangle. Her repetitive use of the "So" with the appropriate intonation suggested that whatever came next was an inevitable entailment of the "one, two, three" sequence. The routine is presented schematically in Figure 5 along with four implementations we witnessed in the span of less than 10 min. The successive performances reveal students' growing command of the routine. Their familiarity with the routine course of actions expresses itself, among others, in the fact that they were able to anticipate the teacher's forthcoming moves and could overtake her role. The teacher, on her part, was happy to be able to gradually withdraw her scaffolding.

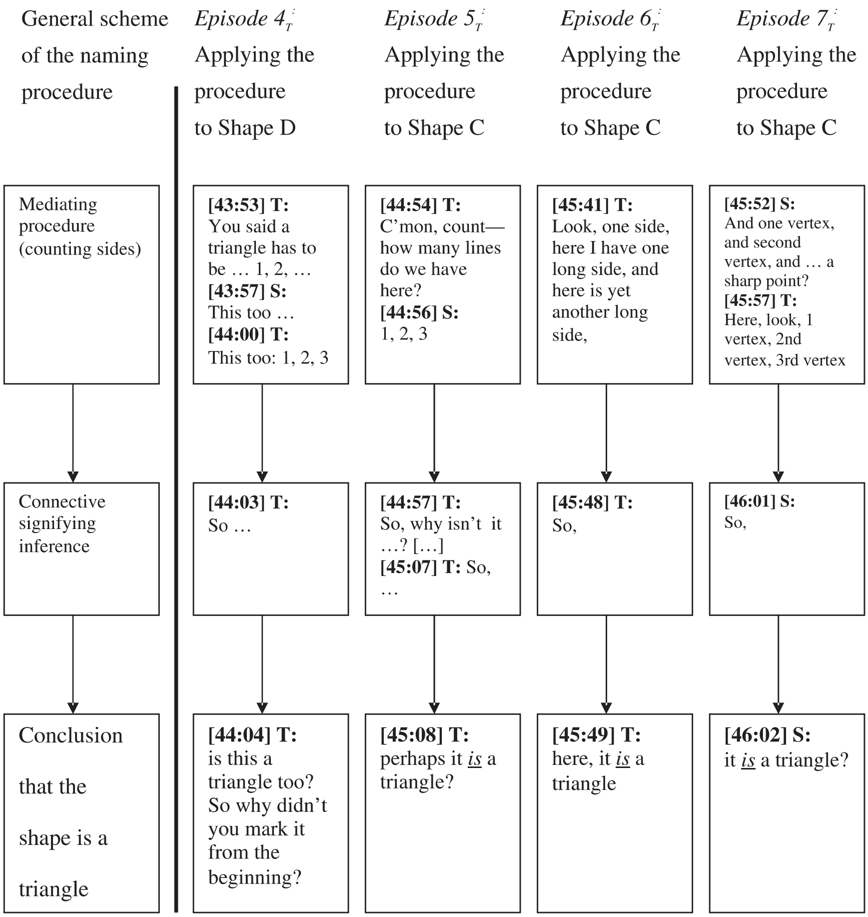

FIGURE 5 Four decreasingly scaffolded performances of the mediated identification routine. T = teacher; S = student.

Focus on Results: Has the Expected Change Occurred?

To recap, the teacher's method of repeated use of a certain discursive sequence, although not immediately effective, did have an impact: Eynat appeared to accept the teacher's discursive perspective almost from the outset, and even her more hesitant friend Shira soon learned to complete the procedure of counting the three sides with the words "So, this is a triangle." And yet, the fact that in the case of "the stick" (Shape C) Shira uttered this conclusion as a question rather than as a firm assertion signaled that she was deferring to the expert rather than acting out of a true conviction. Upon closer look, a trace of doubt could be seen also in Eynat's discursive actions. Thus, even if by the end of the learning session the girls seemed to have modified their use of the word *triangle*, we have a reason to suspect that the change was not yet deep enough. This suspicion was reinforced during the next meeting, when the children were asked to distinguish between rectangles and other polygons. Just like the word *triangle* was ruled out as an additional name for the stick, the word *rectangle* was vehemently rejected as a second name for the square. The following discussion took place as a result:

Episode 8_T: Square that refuses to be a rectangle

[50:20]	Teacher:	Hold on, isn't square a rectangle?
[50:25]	Eynat and Shira:	No!
[50:27]	Teacher:	Why?
[50:31]	Eynat:	Because it is shorter.
[50:33]	Shira:	Because this is like this [*draws a square*] and rectangle is like this, longer [*draws a rectangle with the horizontal side clearly longer than the vertical*].
[50:40]	Teacher:	What did we say? … Eynat, why can't it be … What is it?
[50:43]	Eynat:	Because it is short.
[50:46]	Teacher:	What do you mean?
[50:49]	Eynat:	Short like a midget.

Summary of the Commognitive Interpretation of the Process of Learning About Triangles

According to commognitive interpretation, the transition from the level of analysis to the level of abstraction in geometrical discourse requires insertion of discursive actions into the activity of identifying, which, so far, has been based on direct perception. When this change happens, the use of words such as *triangle* or *square* is no longer a matter of a single-step action "imposed by the world itself;" how these words should be used is now conceived as determined by the users. The mediating discursive routine drives a wedge between the formerly undistinguishable acts of recognition and naming and, as a result, the act of identification is now split into a series of autonomous steps.

In our study, the learners and the teacher, although exposed to the seemingly unbridgeable gap between their respective uses of the word *triangle*, did not realize that this difference was a result of the disparity in their identification routines. In other words, they were unaware of the incommensurability of their discourses and of the commognitive conflict they were facing. This is probably one of the reasons why the teaching–learning process did not seem to be fully effective: By the time the study ended, the children had not yet made the full transition to the new identification routine. This said, they did seem to have made a certain progress, if only because their confidence in the direct identification had been visibly shaken. Problematization of what had previously been taken for granted and was never explicitly considered is, indeed, the first step toward any change.

OBJECT-LEVEL REFLECTION: THE IMPORTANCE OF LEARNING–TEACHING AGREEMENT

On the face of it, the two stories of learning told in this article have disappointing endings. Although in both cases the presence of a commognitive conflict created substantial opportunities for learning, in neither of them did we see students completing the expected transition to a new, qualitatively different set of metadiscursive rules. In the case of negative numbers, the teacher was also visibly displeased with the course of events in her classroom, and she repeatedly expressed her discontent not just with the students' progress but also with herself. In this concluding section I wish to make an argument against the claims about the failure of the learning–teaching processes. According to commognitive interpretation, the reluctance we saw in the students is rather inevitable. At the critical developmental junctures involving new meta-rules, the required change will not happen without a struggle. In the cases under study we might have simply not allowed enough time for this struggle to end. This said, it is also quite clear that at least in the case of negative numbers, there was room for pedagogical improvements. Below I elaborate on all these points by (a) examining in more detail the challenges of major discursive transitions involving changes in meta-rules, (b) addressing the question of what transforms commognitive conflict from an obstacle to communication into a genuine opportunity for learning, and (c) returning to the two teachers who participated in our studies and asking what (if anything) they could have done differently.

Inherent Circularity of Discursive Change as the Principal Challenge for the Learner

Commognitive conflict, although often indispensable for a discursive change, is also potentially dangerous. Although usually invisible and easily mistaken for a factual disagreement, it may eventually stymie further communication. The process of overcoming the conflict is complicated by a certain inherent circularity. To implement the complex change in the unwritten rules of the game, the children must have a good reason. The most powerful driving force is the awareness of the necessity of the required change or at least of its prospective gains (here, *necessary* means imposed from outside, by the world itself). And yet in our study, neither a necessity nor the expected usefulness seemed to be likely incentives for children's learning. As stressed many times along these pages, the new metadiscursive rules introduced by the teacher were not a result of deduction. Although mathematicians have found the resulting discourse useful, the young learners had no means to envision and assess the value of this discourse before actually gaining some experience in applying it in problem solving.

This somehow paradoxical, indeed impossible, nature of the discursive change has many pedagogical implications, one of them being that we cannot expect the difficult transitions to happen rapidly, in a single decisive step. In those special cases when the change is in the most tacit of meta-rules, there is a need for a delicate interplay of getting used to and of making sense of the new game. In this intricate process, time seems to be one of most important factors. Thus, even if in these studies we were not able to see the desired results, some learning was, in fact, taking place, except that the period of observation was too short to show this complex process come to the fruition. If we could revisit the students after a year or two, the odds are that they would tell us what Stendhal eventually told himself: "It *must* be that minus times minus must be plus. After all, this rule is used in computing all the time and apparently leads to true and unassailable outcomes" (quoted in Hefendehl-Hebeker, 1991, p. 27).

Learning–Teaching Agreement as a Condition for Learning

Being a matter of social agreement rather than of natural necessity, the routines of mathematizing can only become one's own through one's peripheral participation in their collective implementations. From being a mere bench player, the learner proceeds to a scaffolded participation. Over time, he or she lets the scaffolding be gradually removed. To quote James Gee,

> Discourses are not mastered by overt instruction (even less so than languages, and hardly anyone ever fluently acquired second language sitting in the classroom), but by enculturation ("apprenticeship") into social practices through scaffolded and supported interaction with people who have already mastered the Discourse.[26] (Gee, 1989, p. 7)

Echoing Gee's observation on the centrality of "people who have already mastered the Discourse," I now wish to argue that proactive participation of the expert interlocutors is critical to the success of learning—and the stress in this last sentence is on the word *proactive*. Indeed, commognitive conflict, and thus a true opportunity for learning, is most likely to arise in a direct encounter between differing discourses. This said, it is reasonable to assume that some special conditions must be fulfilled to ensure that such an encounter catalyzes the desirable change rather than remains an insurmountable hurdle to communication. The required change of the newcomers' discourse seems unlikely to occur without a *learning–teaching agreement*—without a certain set of unwritten understandings about

[26]In Gee's writings, the term *Discourse*, with the capital *D*, has a special meaning that is not far from the one the word *discourse* has within commognitive framework.

those aspects of this process that are essential to its success. Indeed, all the participants need to be unanimous, if only tacitly, about at least three basic aspects of the communicational process: the *leading discourse*, their own respective *roles*, and the *nature of the expected change*. Let me elaborate on each of these requirements.[27]

Agreement on the leading discourse. In the case of commognitive conflict, interlocutors face two conflicting discourses. It is clear that the conflict will not be resolved if each of the participants goes on acting according to his or her own discursive rules. Agreement on a more or less uniform set of discursive routines is the condition for effective communication. Although this agreed set of rules will be negotiated by the participants and will end up being probably somehow different from each of those with which each individual entered the interaction, the process of change may be ineffective if the interlocutors do not agree about which of these initial discourses should be regarded as setting the standards.

The issue of leadership in discourse is, of course, a matter of power relations. In a traditional classroom, the power structure was supposed to be fully determined by the institutional context, and the discourse of teachers and of textbooks counted, by default, as the leading form of communication. According to the principles of reformed child-centered curricula (see, e.g., *Principles and Standards*; National Council of Teachers of Mathematics, 2000), the power relations in mathematics classroom should now be subject to negotiation. In particular, the leadership in discourse is supposed to be attained through agreement rather than by means of imposition—the leader should be accepted and understood, not just mindlessly obeyed. To retain his or her role as a leader without compromising students' agency, the teacher needs to be trusted, and the discourse he or she represents must be valued (e.g., because being an insider to this discourse is considered to be socially advantageous).

Agreement on interlocutors' roles. Once the choice of the leading discourse is made, those who are given the lead must be willing to play the role of teachers, whereas those whose discourses require adaptation must agree to act as learners. The acceptance of roles is not a formal act. Rather than expressing itself in any explicit declaration, this role taking means making a genuine commitment to the communicational rapprochement. Such an agreement implies that those who have agreed to be teachers feel responsible for the change in the learners' dis-

[27]The notion of learning–teaching agreement can be seen as a communicational counterpart and elaboration of Brousseau's idea of *didactic contract*, that is, of "the system of [students' and teachers'] reciprocal obligations" (Brousseau, 1997, p. 31). I do not claim that learning–teaching agreement is sufficient for success in overcoming the commognitive conflict—I only say it is crucially important for learning. This is a theoretical assertion, analytically derived from basic tenets of my approach. The findings in the present study seem to corroborate this claim.

course, and those who have agreed to learn show confidence in the leaders' guidance and are genuinely willing to follow in the expert participants' discursive footsteps (as documented in research literature, cases of students' resistance are not infrequent these days; see, e.g., Forman & Ansell, 2002; Litowitz, 1997). Once again, it is important to stress that this acceptance of another person's leadership does not mean readiness for mindless imitation. Rather, it means a genuine interest in the new discourse and a strong will to explore its inner logic.

Agreement on the necessary course of the discursive change. Agreeing about the discourse to follow and the readiness to shape one's own discourse in its image are important factors in learning, but it is not yet clear how children can possibly "bootstrap" themselves out of the circularities inherent in commognitive conflicts.[28] Upon closer look, it seems that they have no other option than to engage in the leading discourse even before having a clear sense of its inner logic and of its advantages. As has been repeatedly emphasized, awareness of the gains can only be acquired through participation. At this initial stage, children's participation is possible only if heavily scaffolded by expert participants. For some time to come, the child cannot be expected to be a proactive user of the new discourse: In his or her eyes, this form of talk is but as a *discourse for others*, that is, a discourse that is used for the sake of communication with those to whom it makes sense. The goal of further learning is to turn this discourse into a *discourse for oneself*, that is, the type of communication in which the person is likely to engage on his or her own accord while trying to solve his or her own problems.[29]

To sum up, students' persistent participation in mathematical talk when this kind of communication is for them but a discourse for others seems to be an inevitable stage in learning mathematics. If learning is to succeed, all participants (the students and the teachers) have to have a realistic vision of what can be expected to happen in the classroom. In particular, all the parties to the learning process need to agree to live with the fact that the new discourse will initially be seen by the participating students as somehow foreign, and that it will be practiced only because of its being a discourse that others use and appreciate. Let me stress that the exhortation to involve the student in other people's discourses is not an attempt to capitalize on the students' well-known, and commonly disparaged, wish to please the teacher. Entering into foreign forms of talk (and thought) requires a genuine interest and a measure of creativity. To turn the dis-

[28]For an interesting exchange on bootstrapping in learning (known also as *Baron von Münchausen Phenomenon*) see Volume 8 (2001), Issue 1, of the journal *Mind, Culture, and Activity.*

[29]The term *discourse for oneself* is close to Vygotsky's idea of *speech for oneself*, introduced to denote a stage in the development of children's language (see, e.g., Vygotsky, 1987, p. 71). One should also mention the analogy with the Bakhtinian distinction between *authoritative discourse*, a discourse that "binds us, quite independently of any power it might have to persuade us internally," and *internally persuasive* discourse, one that is "tightly woven with 'one's own world'" (Bakhtin, 1981, pp. 110–111).

course for others into a discourse for oneself, the student must actively explore other people's reasons for engaging in this discourse. This process of thoughtful imitation seems to be the most natural way, indeed, the only imaginable way, to enter into new discourses.[30] It is driven by the need to communicate, the need so strong that it would often lead to what may seem in the eyes of some educators as the reversal of the proper order of learning: The learners accept a rule enacted by another interlocutor as a prelude to, rather than a result of, their attempts to figure out the inner logic of this interlocutor's discourse. Without the overpowering urge to communicate and the resulting readiness for the thoughtful imitation, we might never be able to learn anything that is uniquely human—not even our first language.[31]

Back to the Empirical Studies: What Could the Teachers Do Differently?

The natural thing to do now is to examine the status of the learning–teaching agreement in each of our two studies. In the case of the first graders learning about geometric figures, this commitment seemed to be in place and robust: Although not yet sure of the teacher's message, the girls were clearly willing to listen and to look for the inner logic of her actions. The teacher, in turn, although exasperated with the children's stubborn disrespect for the definition, was fully determined to change what needed to be changed; so determined, in fact, that she did not mind repeating the same sequence of actions over and over again, without ever getting tired. The situation was somewhat different in the case of the seventh graders grappling with the negative numbers. Here, the teacher decided to withhold her discursive initiatives and to request that the students discover the rules of the new form of communication for themselves. In this manner, the teacher inadvertently violated not only the third component of the learning–teaching agreement (agreeing on the way to proceed), but also the first (the principle of choosing and following a leading discourse). This decision can be interpreted as an unrealistic attempt to spare the children the experience of discourse for others altogether. The teacher's subsequent refusal to demonstrate her own discursive ways left the children without a

[30]As Vygotsky (1978) reminded us, a sociocultural vision of learning (and, in particular, his own notion of zone of proximal development) must result in "reevaluation of the role of imitation in learning" (p. 87).

[31]Let me add that we often insist on children's own inventions not only because of the learning opportunities they create, but also because we believe that in this way we show more respect to the learner and, while doing so, help him or her to be a better person. However, belief in the possibility of children's unmediated learning from the world and the wish to sustain the democratic spirit of the classroom discourse are not an indissoluble whole, and abandoning the former does not necessitate compromising the latter. It seems that realistic learning–teaching agreement can be cultivated in schools without any harm to the democratic spirit of classroom interactions. Indeed, making sense of other people's discourse is not any less creative or demanding than "reading the codes" of nature.

clue about where to look for the leading discourse.[32] In the thusly created leadership void, the children chose to follow the discourse of the person who was known as a leader of many other discourses. The teacher's reticence had an unhelpful impact also on the students' further learning. Leadership once renounced can not be easily be regained. When the teacher decided to be more explicit about the new meta-rules, the children greeted her attempts with disbelief. They were no longer taking the superiority of teacher's discursive ways for granted. They became suspicious of what they could not understand quickly.

The seventh-grade teacher should not have given up her leadership so easily. As theoretically argued and empirically reinforced, it was unreasonable to expect that the students would make their own decisions before the relevant new routines had been demonstrated and before the criteria for accepting or rejecting different options had been negotiated and agreed upon. Indeed, the principle of giving agency to students does not mean withholding the teacher's own. Quoting Magdalene Lampert, introducing students to mathematical discourse is "like teaching someone to dance": It necessarily involves "some telling, some showing, and some doing it [by the teacher] with [the students], along with regular rehearsals" (Lampert, 1990, p. 58). This said, cultivating learning–teaching agreement in a classroom is a delicate matter. For one thing, the institutional context of school blurs the difference between the pursuit of democratic leadership and an attempt to force dominance. What is intended by the teacher as a plea for confidence may be interpreted by some students as an attempt to exert power. In addition, whether the teacher is accepted as a leader or not—whether she is trusted and her discourse is valued—is not just a simple function of what happens in school. When it comes to issues of alignment versus resistance, cultural factors may be of principal importance. All too often, classroom norms that seem most conducive to mathematical learning remain in conflict with the norms of the outside world.[33]

Another thing mathematics teachers can do to support students in turning the discourse for others into discourse for themselves is engage the students in an ongoing conversation about sources of mathematics. Much can probably be attained by putting human agency back into the talk about mathematical objects—by making it clear that mathematics is a matter of human decisions rather than of externally imposed necessity. On this occasion, the importance of custom and learning–teaching agreement should be brought to the fore.

[32]This observation is in tune with the one made by Nathan and Knuth (2003) in their analysis of one teacher's practices: "Students … may not and often cannot serve as the analytic authority necessary to promote correct understanding about all the content matters" (p. 203).

[33]These days, there are signs that students' recognition of the value and importance of mathematical discourse cannot be taken for granted (Sfard, 2005). Also, the stress on the learner's agency may sometimes be seen as remaining in conflict with the requirement of alignment with somebody else's discourse.

Aware of the social origins of the rules and objects of the discourse, the student will have a better sense of where to turn to while searching for answers to certain questions. Yet another potentially helpful, even if not easy, didactic move, at least in the case of older students, is to make the commognitive conflict into an explicit topic of conversation. Of course, explicating metadiscursive rules is not an easy task, and, even if successfully implemented, it does not ensure that the required change will be accepted smoothly and without protests. Still, having a debate is certainly preferable to remaining silent about what needs to be done.

METALEVEL REFLECTION: WHAT COMMOGNITIVE LENS HAS REVEALED AND WHAT STILL WAITS TO BE SHOWN

It is time now to ask whether the commognitive framework fulfilled our hopes and proved itself as a conceptual lens strong enough to let us deal with technology-enabled high-resolution portrayals of learning–teaching processes. To answer this question, let me list some of the potentially generalizeable insights we were able to gain in our two studies.

The notion of commognitive conflict proved useful in the attempt to understand certain particularly resilient difficulties experienced by students at certain well-defined points in curricula. In our two studies, the fine-grained commognitive analyses revealed that, at least in some cases, these difficulties may stem from the necessity of a transformation in metadiscursive rules. These rules usually remain tacit. Their transformations, rather than being a matter of a logical necessity, are a result of a change in customs. In view of this, it is no longer surprising that at these special discursive junctures the majority of learners tend to falter. As an aside, let me remark that this observation, although made in classrooms, may be useful also in analyzing the historical development of mathematical discourses. The same commognitive conflict that hindered our seventh graders might have been responsible for the difficulty experienced by 16th, 17th, and 18th century mathematicians. The tug of war between incompatible metalevel rules seems to define property of those historical events that post-Kuhnian historians of mathematics call *mathematical revolutions* (Kitcher, 1984).

A direct didactic implication follows from this latter observation: Although at first sight the results of the teaching–learning effort observed in our studies were not fully satisfactory, the teachers should not despair. As implied by the commognitive analyses, the difficulties revealed on these pages, rather than being an unintended result of particular instructional approaches, were part and parcel of the process of learning. These difficulties were to the change in discourse what friction is to the change in movement: the necessary condition for such change to occur. To successfully cope with the friction, all the parties involved— the teacher and the students—needed to be truly committed to the success of communication.

And yet the unwritten agreement about the way in which the gap between their present discourses should gradually be closed was not always honored.

The results of this interim stocktaking may suffice to explain why I feel that the commognitive perspective does seem to fulfill its promise. It ushered me and my coresearchers into hidden strata of learning–teaching processes. The final accounts of what we saw seem to me more helpful, than the ones I would have been able to construct a few years ago while working within a more traditional framework. Among the gains let me count our present awareness of the difference between object-level and metalevel learning; our current understanding of this latter type of learning, the special difficulty of which is due, among other things, to the tacit nature of the change and to the contingency of the meta-rules that are the object of change; and our ability to draw novel conclusions about conditions for learning and then to follow the nontrivial pedagogical implications of these conclusions.

This said, I am aware that much conceptual and methodological work is yet to be done. One of the new questions our research team is now facing as a result of our commognitive analyses regards the ways in which to foster learning–teaching agreements in mathematics classrooms. Considering the fact that such agreements depend on values and norms of the wider society, this question is definitely not easy to answer. Our present hunch is that we should try to help ourselves with the notion of identity, which has already drawn the attention of some researchers and which seems particularly appropriate for the role of the conceptual link between theories of culture and those of learning (Gee, 2001; Gutiérrez & Rogoff, 2003). Although the notion is yet to be defined in operational terms (for the first commognitive attempt at operationalization, see Sfard & Prusak, 2005), a general agreement seems to exist about a couple of basic traits that need to be captured by the definition. These consensual properties of identities—their power to shape human actions and their location at the crossroads of multiple discourses—make the notion of identity naturally predestined for the role of a conceptual linkage between the general and the specific, between the collective and the individual. If carefully defined and immunized against essentialist interpretations, this notion may thus help in explaining how intradiscursive forces shape individual discourse. Because all these ideas are yet to be developed, I do not expect the sign "under construction" to be removed from the commognitive framework any time soon.

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