# Fluid Solutions

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## 1 Pressure and Buoyancy

1. When completely submerged, the cup displaces about $\pi r^2 h$ water, which has a mass of $\rho\pi r^2 h$. If the cup's mass is $m$, then the net upward force is the buoyant force minus the cup's weight, or

   $$F = (\rho\pi r^2 h)g - mg$$

   Therefore you have to exert this much force downward to hold the cup in place.

2. The pressure at the bottom of the bowl is $\rho g H$. Since pressure must be the same in all directions, the air pressure inside the cup must be the same.

3. Interestingly, these two things are not related. This makes sense when you think about it – if the cup were much heavier, it would sink, but the pressure inside would remain the same. Also, if you push the cup farther down into the water, the pressure inside will increase, even though you do not have to push harder to hold the cup.

4. When you drop a rock in the bowl, the water level rises to some $H' > H$. Therefore the pressure increases, to $\rho g H'$.

5. The buoyant force on the cup does not change, since buoyancy is independent of depth.

6. 
   - Below the surface: $F = \rho g V - mg$.
   - Partly submerged, the cup displaces a volume equal to the amount of cup still submerged. So if $h'$ of the cup remains beneath the surface, the upward force is $F = \rho g(\pi r^2 h') - mg$.
   - Above the water, $F = -mg$.

## 2 Leakage

The general idea here is to use both conservation of energy and conservation of momentum. From conservation of energy, we can find out how fast the jet will be going when the potential energy of water pressure is converted to the kinetic energy of the moving jet. So we know how fast the water is moving, but not how much of it there is.

To figure this out, we use conservation of momentum to figure out how much momentum comes out of the hole due to the force of water pressure. In other words, we know how fast the stream is going, and how hard it's being pushed. We want to find out how much water can be made to go that fast using exactly that amount of force. Since momentum is mass times velocity, and we now know the velocity and momentum, we can calculate how much mass (hence how much volume) goes through the hole, and therefore figure out the size of the stream.

1. The static pressure is $\rho gh$.

2. The potential energy per mass due to pressure at depth $h$ equals the difference in gravitational potential energy between the surface of the water and depth $h$, or $gh$. This is true because the total potential energy from gravity and pressure must be the same at every point in the fluid – moving one piece of water to another place won't change the amount of potential energy in the water.

3. Assuming the pressure in the jet of water is zero, the potential energy from pressure will all have been converted to kinetic energy ($KE = 1/2mv^2$). So using conservation of energy (per unit mass),

$$\frac{1}{2}v_{out}^2 = gh \implies v_{out} = \sqrt{2gh}$$

4. Pressure is force per area, so the total force on the hole is the pressure at its depth times its area, or $\rho ghA$

5. Momentum is mass times velocity. We know the jet's velocity from above, and we can calculate the mass leaving the hole in time $t$ as density times the volume passing through the hole in $t$. Since this volume equals the area of the hole, times the "length" of jet passing in $t$ (velocity times $t$),

$$m = \rho A' v_{out} t$$

So the total momentum leaving the hole is

$$m v_{out} = \rho A' v_{out} t v_{out} = \rho A' v_{out}^2 t$$

6. Force is momentum per time, so we can combine our expressions for force (pressure times area) and momentum from above:

$$\begin{aligned} \rho A' v_{out}^2 &= A\rho g h \\ \Longrightarrow A' &= \frac{gh}{v_{out}^2} A \\ &= \frac{gh}{\sqrt{2gh}^2} A = \frac{A}{2} \end{aligned}$$

So the area of the jet is exactly half that of the hole. Strange...

# 3 Beading water on a spiderweb

Assume that a thread of length $L$ is covered with a uniform layer of water of radius $r$. Then its volume is $\pi r^2 L$, and its surface area is $2\pi r L$.

Now imagine that this water is gathered into $N$ drops (each $d = L/N$ apart), each with radius $R$. Then the volume of these drops is $N\frac{4}{3}\pi R^3$, and their surface area is $N 4\pi R^2$.

To solve for $R$, we remember that the volume of water is conserved, so

$$\pi r^2 L = \frac{4}{3}\pi R^3 \frac{L}{d} \Longrightarrow r^2 = \frac{4}{3} R^3 \frac{1}{d} \Longrightarrow R^3 = \frac{3}{4} d r^2$$

Let's leave volume alone for the moment, and take a look at surface area.

For the drops to remain drops, they must have a lower surface energy than the cylinder. Therefore they must have a smaller total surface area, or

$$\begin{aligned} 2\pi r L &> 4\pi R^2 \frac{L}{d} \\ \Longrightarrow rd &> 2R^2 \end{aligned}$$

Now we can put in our expression for $R$ from above:

$$\begin{aligned} rd &> 2\left(\sqrt{3}\frac{3}{4} d r^2\right)^{2/3} \\ \Longrightarrow r^3 d^3 &> 8\left(\frac{3}{4} d r^2\right)^2 \\ \Longrightarrow r^3 d^3 &> 8\left(\frac{9}{16} d^2 r^4\right) \\ \Longrightarrow d &> \frac{9}{2} r \end{aligned}$$

So the drops have to be at least $\frac{9}{2}r$ apart.