Learning Math as Ontological Change

Abstract

If mathematics is a sociocultural activity, learning math involves “socialization.” We analyze a fifth grade math lesson in fractional equivalents, and show that the lesson is not about constructing knowledge so much as about producing and acting on new species of mathematical object: “fractions.” As the children learn to recognize and act appropriately on these objects they too, we propose, are ontologically changed. We suggest that classroom academic tasks have an embedded cultural task. While school uses social interaction to achieve academic ends, at the same time academic tasks are used to achieve cultural ends. The children are not just learning math, they are learning to be a particular kind of person....
“By seeing math and science not as disembodied pieces of knowledge or skill but as the inventions and discoveries of particular people, as products of their hopes and disappointments, their struggles and problems, we can begin to re-embed those subjects again in their proper human contexts, in which they initially had affective, as well as purely cognitive, meaning” (Egan, 1997, p. 64).

Introduction
A decade of debate has not yet resolved the issue of whether the best explanatory framework for studying existing mathematics instruction, and designing new instruction, is constructivist or sociocultural, or some reconciliation of the two (see Cobb, 1994; Donmeyer, 1996; Greeno, 1997; Hiebert, 1996; Packer & Goicoechea, 2000; Salomon, 1995; Sfard, 1998). The debate is made more complex by the fact that both the constructivist and sociocultural approaches have several variations, and because the differences between them have been characterized in a variety of ways.

The variations in constructivism stem from differential emphases on Piagetian work or on cognitive science, from reinterpretations of the constructivist program (such as von Glasersfeld’s “radical constructivism”), and from formulations of a social constructivism (e.g. Doise et al., XX). The variations in socioculturalism stem from different interpretations of Vygotsky, from the different disciplines (psychology, anthropology, etc.) that socioculturalists were trained in, and from the disagreements natural to an emerging paradigm.

The differences between constructivism and socioculturalism have been characterized in a variety of ways. Some have argued that socioculturalism focuses on social phenomena and neglects the individual (e.g., Hiebert, 1996), while counterarguments have been made that constructivism cannot explain social life and “intersubjectivity” in the classroom (cf. Thompson & Steffe, 2000; Lerman, 1996; 2000). Socioculturalism has been also described as viewing the child as a passive recipient of knowledge presented by the teacher, a figure of authority, while constructivism attends to the child’s active role in the construction of knowledge (e.g., Cobb et al., 1997; Cobb & Yackel, 1998), a charge which has been denied (e.g., Washescio, 1998). And socioculturalism has been characterized as attending to practical activity, to social context, and neglecting the mental activity, the cognition, that constructivism is attuned to (XXX).

Even when the aim has been to broker a reconciliation of the two approaches rather than to reject one or the other completely, or to draw on both in order to conduct “complementary” analyses (e.g., Cobb, 1994; Greeno, 1997; Wertsch & K, in press), there has been a tendency to apply a set of related dichotomies: individual/social; active/passive; mental/manual; thought/action. In our view the dichotomous characterizations in these “paradigm wars” (Forman, Sfard & ?) do not stand up to careful scrutiny, at least as attributes of the two frameworks in general. Though particular formulations may fall into one or another of these errors, socioculturalism in general has the merit, we have argued (Packer &
Certainly Vygotsky himself saw beyond these binary distinctions. Rejecting the dualism of individual/social, he wrote that “[t]he individual personality… is the highest form of sociality” (Vygostky, 1986, p. 54, cited in Yoroshevsky, 1989, p. 219). It is apparent that for Vygotsky “the individual psyche is a recent historical product which has emerged from the depths of collective life, a product that bears an indelible imprint of the processes taking place in these depths” (Yaroshevsky, 1989, p. 220). Far from seeing learning as a passive internalization of knowledge he wrote that “from the scientific point of view… the assumption that the student is simply passive… is the greatest of sins, since it takes as its foundation the false rule that the teacher is everything and the student nothing” (Vygotsky, 1997, p. 48). And far from practicing socioculturalism as a sociological analysis of cultural phenomena that disregards psychological phenomena such as thought and cognition, Vygotsky’s aim was to chart the dynamic and mutually constitutive interplay of these two, such as “the regulation of human behaviour by a system of higher psychical functions constructed with the aid of objective ‘things’ of culture” (Yaroshevsky, 1986, p. 257).

At the same time, as Cole and the Laboratory ?of Comparative Human Cognition have pointed out, mental operations are “hypothetical processes [never] directly available to the analyst. Rather, processes are inferred from differences between two or more task environments” (LCHC, 1978, p. 54). Even for constructivism it is theoretical reconstruction, sometimes as computer simulation, sometimes as mathematical models, rather than direct observation, that “becomes the guarantee for the reality of the psychological processes said to be inside the head” (p. 54).

LCHC (1979) also note, citing Greenfield (1976, p. 327), that “although the role of the organism-environment interaction is central to his constructivist theory, Piaget has never specified the nature of these interactive processes nor has he himself made them the object of empirical study.” Contemporary constructivists have long since remedied this omission, but it is consequently ironic that they would charge socioculturalists with neglecting cognition by focusing on interaction. The issue is not whether to study classroom interaction or individual cognitive activity. Since mental processes are invisible, both constructivists and socioculturalists must be content with studying activity in concrete settings. The key issue is how best to interpret classroom interaction, in order to appropriately ground inferences about its products, whether these be ‘in the head’ or ‘in the world,’ or both.

The Ontological Level
In our opinion, some of the confusion over the similarities and differences between constructivism and socioculturalism stems from a neglect of the ontological commitments in each approach, for it is precisely here that is found the source of dualism (in constructivism) and its avoidance (in socioculturalism). A consequence of this neglect is that socioculturalism is interpreted as a merely another theory of the construction of knowledge: albeit one that emphasizes the social rather than individual nature of this construction, the way cultural artifacts mediate organism-environment interaction, and how interpersonal activity is ‘internalized’ as intramental activity. But constructivism accepts a Kantian
ontological distinction between objects as we represent them and objects as they are in themselves, and focuses on the way mind structures knowledge, while Vygotsky’s socioculturalism builds instead on a Marxist-Hegelian philosophy of the mutual constitution of human and (social) world (Packer & Goicoechea, 2000). Piaget presumed a natural ontological distinction between known world and knowing ("epistemic") subject. For Piaget the epistemic subject is unchanging, and objects are unchanged by its operations (they are merely displaced); this was the basis for Vygotsky’s criticism of Piaget for a “subjective idealism” (p. 50) in which “external reality plays no substantial role in the development of a child’s thought” (p. 48); that one of the “central flaws in Piaget’s theory” is that “it is reality and the relations between a child and reality that are missed” (pp. 51-52).

Vygotsky, in contrast, investigated mutually constitutive linkages between subject and object, child and world, thinking and speech (cf. V 1986, p. 1-11). His socioculturalism considered not only knowledge to be constructed, but also the objects known and the people who know them. Objects are not naturally occurring, but are produced and transformed by human activity, constituted by cultural practices – this is one reason for the emphasis on ‘artifacts’ rather than simply ‘objects.’ Individuals too are not natural but products of culture and history. The “epistemic subject” is not a fixed and unchanging essence, but is fluid and transformed. It follows that learning involves ontological change.

The different ontologies of socioculturalism and constructivism are relevant to educational research because teaching is not just about building children’s knowledge, but also about influencing the kind of person they become (Dewey?). Teaching math is not simply about learning math! Once we recognize this it is easier to understand the heated debates over math curriculum and ‘fuzzy’ versus ‘traditional’ approaches to math instruction, because they are conflicts not simply about the best way to learn math, or even about the best math to learn, but about the appropriate attitudes and values that schools should foster in young people: autonomy or interdependence, obedience or creativity? There is room in this debate for valid disagreement, for it requires a solution that is political, not technical.

This is not to say that research has nothing to offer. But what is needed is research attentive to the ways history and society shape the course of development (“quote”), that is to say to ontological processes, not just epistemological ones. This, we believe, is a kind of research possible within the sociocultural but not the constructivist framework: research that investigates learning as entailing transformation both of the person and of the social world – and the way these changes are inextricably interrelated.

**Studying Mathematics**

Our aim in this paper is to demonstrate a sociocultural approach to the study and analysis of learning that attends to these ontological as well as epistemological phenomena, that attends to construction in classroom interaction, and also to cultural and historical context. We intend to illustrate the power of an analysis of the “ontological work” of learning, by examining an episode of mathematics instruction that dealt with fractional equivalence, in a 5th grade classroom.

The practice and learning of mathematics have a special place in both sociocultural and constructivist research. After all, *mathesis* is learning. As Forman (1999) points out, “For the past 40 years,
cultural psychologists have learned a great deal from their studies of mathematics learning. Much of the cross cultural research that formed the theoretical and empirical basis for cultural psychology focused on mathematical activities... and the cross-fertilization continues in both directions.” And recently the study of mathematics education has been the occasion for a skirmish in the “paradigm wars” (Forman et al.), as core concepts in sociocultural and constructivist theories are applied and tested.

To the sociocultural approach, the study of mathematics is an important test case. Mathematical concepts and operations were from the start the core focus of Piagetian research, and if the sociocultural approach is to offer an alternative or corrective to constructivism it must do so here. In particular, if we are to demonstrate that learning has an important and neglected ontological level, the study of learning mathematics is important. Above all, it is not immediately obvious that mathematics has an ontological level; it appears to be a purely epistemological enterprise. Mathematics seems an activity conducted entirely ‘in the head,’ dealing with a pure kind of knowledge, atemporal truths that have nothing to do with culture. Mathematicians are typically considered dispassionate intellectuals, dreamers, unworldly loners, solitary thinkers, creative geniuses. So in the next section we seek to show that there is an ontological aspect to mathematics, visible once its history is investigated. It turns out that mathematics involves fundamental – and changing – ontological commitments.

The subsequent section offers our detailed exegesis of a lesson on fractions and equivalence. We examine the ontological work accomplished in the classroom discourse – work that is social, interactive, based in the pragmatics of indexical reference, in conversational implicature, and the contextuality and historicality (in this case, primarily micro-historicality) of discursive action. We intend to illustrate that it is possible to understand teaching-and-learning as a transformative practice in concrete terms, using publicly-available evidence, and invoking the ‘black-box’ of student cognition. We propose that learning math should be understood as a process in which mathematical objects are created, along with correlative knowing subjects. Specifically, we shall describe how the students come to iterate on fractions.

In the final section of the paper, we turn from the production of mathematical objects to consider the production of mathematical subjects …

The Ontology of Mathematics

It might seem odd to suggest that mathematics has an ontology. It seems the prime example of pure knowledge, of a solely epistemological enterprise. But knowledge must be about something, some kind of entity, and even mathematics involves assumptions, often unexamined, about the entities with which it deals. This is not to say that the status of mathematical objects is self-evident, however. As Sfard (1998) puts it, “[t]he common referent of the symbols ‘2/3’ and ‘12/18’ is an elusive entity, the ontological status of which has been puzzling philosophers for ages.”

Lachterman (1989) traces the genealogy of mathematics and shows how classical Greek and modern Cartesian mathematics were dramatically different; he describes these two periods as “turns on an ontological axis” (p. 5). He suggests that the key to the ontological status of mathematical entities is to be found in the historical “ethos” of mathematical practice: the “settled or characteristic ways human beings have of acting in the world” (p. xi), and he traces the changes in these mores and styles with which
people “comport themselves as mathematicians both toward their students and toward the very nature of those learnable items (ta mathemata) from which their disciplined deeds take their name” (p. xi). A brief summary of his main points is important for our subsequent analysis.

Classical Mathematics

The Greeks considered mathematical entities—numbers, shapes, solids—to exist prior to their knowledge of them, in the famous ‘Platonic’ realm. Math was the way of finding these preexisting, underlying forms. The geometry that Euclid (c. 300 BC) presented in his Elements operated within Plato’s ontology of perfect, immaterial, singular forms (such as Triangle), imperfect, multiple, sensible objects, and “mathematicals”—the mathematician’s inscribed figures, such as ABC and XYZ, which were both immaterial and plural (Lachterman, p. 118). Every constructed square, for example, was considered at the same time a unique individual, different from every other square, and yet also representative of the form of squareness, the perfect Square, and so identical with every other square.

In this classical ethos and ontology, student and teacher were not discovering or enacting something new when they constructed a figure, or divided two numbers. They were rediscovering what had been there all along, “reenacting in time what has already been done all along and thus never for the very first time” (p. 121). Mathematical activity was seen as evoking the perfect geometric and arithmetic forms which were the genuine objects of study (p. 120-121).

Modern Mathematics

The radical moderns, of whom Descartes (1596-1650) was preeminent, viewed math, and knowledge in general, very differently. The 17th century is commonly seen as a time of revolution in mathematics, with the invention of analytic geometry, differential and integral calculus, decimal fractions, and more. Lachterman sees in this revolution a change in ethos and ontology. “Construction” became a central concept in modern mathematics, a concept that was at core ontological. Mathematical entities were now seen as having no existence prior to their human construction. Cartesian math was something active and creative, an example of the mind’s essential power of making. Descartes’ approach to geometry illustrates this: he introduced in La Géométrie (1637) a coordinate system – his famous X & Y coordinates – whereby geometrical problems could be expressed as equations involving variables, known quantities, and unknowns. The solution of the problem (finding a locus, for example) amounted to solving – finding the ‘roots’ of – a polynomial equation. Problems of a specific degree of complexity (number of lines, dimensionality of figures) corresponded to equations of a particular degree. Hence there was an order, a seriality, to this method. It was very general, and powerfully iterative – a solution at one degree of complexity provided the basis for a solution at the next degree. In a real sense this approach turned every geometrical problem into the single problem of finding an equation’s roots. The successes of this form of math fed its confident extension widely through accounts of human thinking and learning.

This modern, Cartesian math of formulae and constructed proofs differed in both ethos and ontology from the classical Greek math of demonstration and proof of theorems. Whereas the Greeks
considered mind the mirror of external things and math a way of finding entities such as numbers that already existed, the Cartesians considered mind active and creative, in its essence a power of making. Math was the outward embodiment of this creative power, made visible in the construction of problems (e.g., positing axes, etc.) and of their solutions (drawing up equations). Modern mathematics dissolved the classical distinction between arithmetic and geometry, and in doing so dissolved the perfect Forms that the Greeks had emphasized. Integers became merely a particular type of rational number; square and cube became merely particular cases of constructions in multi-dimensional space. Infinity and infinitesimals were manipulated with ease.

Problems with Modern Mathematics

This modern mathematical ethos, though born in the 17th century, is alive today. (Its influence on the constructivism of Piaget will be apparent; cf. Rotman, 1977.) But it is not without its problems, as well as its successes. The conception of mathematical construction straddled an ontological divide—commonly associated with Descartes—between the subjective (intentionality, the conceptual) and the objective (the referent, the sensible). The concept of construction came to the fore in large part, Lachterman argues, because it promised to be the mediator between reason and reality, mind and mindless nature. As Lachterman puts it, Descartes “most conspicuously exploited the power inherent in the view that symbolization frees us to work ingeniously beyond the boundaries apparently fixed by nature as it is sensuously, premethodically given... while at the same time serving to direct those mechanical operations or movements from which outwardly manifest configurations artfully issue” (p. 125). Construction seemed to bridge or dissolve the gap between subjective and objective; it seemed to solve the problem of the relationship of reason to the ‘real’ world, of the ‘true’. The “secret” of modernity, what it struggled to achieve, in Lachterman’s view, was the “willed coincidence” of human making with truth or intelligibility.

But the promise couldn’t be made good: increasingly this ethos came to deny the reality of the ‘external’ world. To put this another way, Cartesian mathematics, skeptical of perception as a source of genuine knowledge (e.g., Descartes’ criticism of Locke), seemed to offer a powerful illustration of the power of ‘reason’ (individual mental capacity) to construct truths about the world. But a tension developed. On the one hand, true knowledge could not be based directly on the natural world, for this, in the Cartesian ontology, was a realm of contingencies, while true knowledge was unconditional. On the other hand, reason in general, and mathematics in particular, still strove to be about what is ‘real.’ The criterion of adequacy to reality seemed unavoidable, yet indeterminate.

What had begun as a relatively innocent effort to bring order to the natural world, to “master and possess” it (p. 23), ended with the denial that reality had any existence outside the human mind, a denial that dissolved into relativism and “self-divinization.” For example, when non-Euclidean geometries were developed in the nineteenth century, both these and classical geometry were viewed as abstract systems resting on conventional and arbitrary axioms, rather than either logically necessary or natural postulates. The “reality” of the axioms was deemed irrelevant, and these geometries were considered to offer
conjectural models of physical space (e.g., Einstein’s use of Riemannian geometry), not as descriptions of how space really is.

Postmodern Mathematics

These problems within modern mathematics have motivated several attempts at a new conceptualization of mathematical investigation. Lachterman, for example, can tell his history of ancient and modern math only because he adopts a third position, which one can call postmodern. From this vantage point the problems of modern mathematics become visible.

Rotman (1993), too, seeks to deconstruct the view that math is a purely formal enterprise. In his account, the formal procedures of mathematics – operating on apparently decontextual and abstract entities – are not self-sufficient, but are sustained by the informal practices of a community of mathematicians. “Mathematics is neither a self-contained linguistic formalism nor an abstract game played entirely within the orbit of its own self-referring rules, conventions, and symbolic protocols” (1993, p. 24). Rather, there is a relationship between mathematical formalism and everyday language which “allows an embodied subject – the corporeal, situated speaker of natural language – to register a presence in and connection to the world of real time, space, and physical process” (1993, p. 25). We will return to the character of this “presence” later. Rotman is clear on the need “to demolish the widely held metaphysical belief that mathematical signs point to, refer to, or invoke some world, some supposedly objective eternal domain, other than that of their own human, that is time bound, changeable, subjective and finite, making” (Rotman, 1987, p. 107). Math does refer to a reality, but this is a human reality, itself constructed, neither timeless ideal Forms nor lifeless matter. Mathematical signs circulate within, and help reproduce, a specific sociocultural human context.

Rotman sees modern mathematics as motivated by an “illusion of mastery” that is literally fantastic. For example, the assumption of infinite iterability that one finds in integral calculus and series expansions presumes a counterfactual capacity: such iteration would take infinite time and infinite energy. In its place Rotman proposes a “non-Euclidean arithmetic” in which the iteration that underlies number and counting is not unbounded and infinite, but closed and finite. Such an arithmetic, which Rotman works out in some detail, is only “locally Euclidian.”

Sfard (1998) also gives an account of mathematics that can be called postmodern. She recounts how the search for the elusive referents of mathematical discourse motivated reconceptualizations of this relationship – the move from realism to constructivism, and then the abandonment of the classic dichotomy of symbol/referent in favor of interactionist views of symbols and meaning, such as the semiotics of Saussure and Peirce. Sfard builds on “Foucault’s central claim that the objects ‘referred to’ by symbols, far from being primary to signs and speech acts, are an added value (or the emergent phenomenon) of the discursive activity. This is particularly true for the evanescent objects of mathematics” (p. 14). The “central theme” of her paper is “[t]he process through which the objects ‘represented’ by the symbols come into being retroactively” (p. 15). She suggests that discourse about mathematical referents is “Virtual Reality discourse” rather than “Actual Reality discourse,” a metaphor that “conveys a message as to the particular rights and obligations the mathematical discourse confers.
upon the participants.... Those who really wish to communicate, not being able to help themselves with their senses, have to use all their mental faculties in an attempt to reconstruct for themselves the realm within which the moves of their interlocutors make sense” (p. 2). The task that faces us when we seek to understand mathematics, as she sees it, “consists of not – or no longer – treating discourses as groups of signs (signifying elements referring to contents or representations) but as practices that systematically form the objects of which they speak” (Foucault, 1969/1992, p. 40, emphasis added by Sfard).

These postmodern analyses of mathematics are in agreement that despite its common image as an acontextual and purely formal process of knowing, mathematics is an activity with a history, with an ontology (which has changed), and with an important informal discursive/social/cultural dimension. Rather than seeing mathematics as the prime illustration of the power of the human mind to construct knowledge and thus gain mastery over the natural world, they see math as a particular – and contestable – embodiment of human reason, and of the relations between human endeavor and the natural world. This embodiment has changed, and it can become problematic. The postmodern view of mathematics sees it as a set of signifying practices that produce mathematical objects.

These analyses indicate that to understand how math is taught and learned we need to attend to talk, social interaction, and social practices. When we study children learning mathematics we cannot assume that we (or they) are dealing with a single or well-defined phenomenon. We need to ask what kind of mathematics is being learned, with what ethos, and what ontology. We need to consider the modes of reason, of human activity, and indeed of human being that are brought to life in the curriculum. Thus far “there has been little sustained attempt to develop the philosophical and conceptual consequences of saying what it means for mathematics to be a language or be practiced as a mode of discourse” (Rotman, 1993, p. 17), especially “the question of the being, or rather the becoming, of basic mathematical objects such as numbers and points” (p. 23, original emphasis). Sfard advises that, “For the researcher, the most important implication of these considerations is that, in the attempt to understand people’s thinking, one should focus his or her attention first and foremost on what can be seen ‘with the naked eye,’ that is, on the things people say and write. The objects of the discourse will eventually stand out from the intricate web of utterances just as a three-dimensional shape stands out from an appropriately-inspected stereogram” (page).

How Mathematical Objects Become
Both a broad consideration of the differences between socioculturalism and constructivism, then, and the examination of mathematics in particular, suggest the need to consider the ontological aspects of learning mathematics. In this section we apply and extend our recent reconceptualization of learning and development (Packer & Goicoechea, 2000) through a detailed analysis of mathematics instruction in a fifth grade math lesson on fractional equivalents. It is precisely in “the things people say and write” that we can see “the becoming of basic mathematical objects.” We will describe the production of several kinds of mathematical object, upon which the children learn to act appropriately, and suggest that the
Learning Math as Ontological Change

An Overview of the Lesson

The class we shall consider was in an elementary school in the industrial U.S. midwest, in an ethnically mixed, predominantly working-class, school district, a context we will return to later. The episode of instruction we shall examine in detail began the afternoon, at 1:00 p.m. on a Thursday in May. Our analysis attends closely to the sequential organization of the discourse, which we will sketch in overview before examining in detail:

Each student is given a bag containing colored segments of a circle: light blue wholes, yellow halves, dark blue thirds, white quarters, red sixths, and orange eighths. The teacher directs the following sequence of tasks:

- First, the students make all the possible “wholes” of different colors (transcript lines 56-67).
- Then they find what “equals a whole fraction circle” (line 67), by “dividing” each whole into its parts: they do this in sequence: two halves, three thirds, four quarters, six sixths, eight eighths (67-140). As they do so the teacher inscribes on the board. When they are finished she reiterates what they’ve found: the “equivalents”; the “fractions” equal to one whole (141-155).
- Next they find “what is equal to a half,” by “making” a half with different pieces. Again they do this in sequence: two quarters, three sixths, four eighths (156-202), and the teacher reiterates these “equivalents of one half” (202-226).
- Next they find the equivalents to “one third” (229-287), and then equivalents to “three fourths” (289-331).
- The teacher then asks a student to “give us a fraction” (331) to which they can “find the equivalents”; the student picks three sixths (331-392). This proves more difficult for the students; the teacher notes that there is “a clue” on the board. She asks the students to “go up to” more equivalents to three sixths that they “don’t have,” and once the students respond with twelve twenty-fourths and sixteen thirty-seconds she ends the activity.

Making Wholes

The teacher herself provides the students with an advance overview of the lesson:

T: Okay, I want you to- as you did before, practice a little bit making wholes, and quarters, halves, and then we gonna talk about, adding things and subtracting things and then changing the- uh, mixed fractions and that kind of thing. Now as you make them I want you to notice equivalents, to a whole. I want you to tell me how many pieces make a whole, uh, different shapes are gonna give you different fraction parts.
The activity begins with the students organizing their colored plastic pieces to form complete circles. This serves a preparatory function, to ensure each student has all the necessary pieces, none missing or extra, as well as to clarify that colors should not be mixed and that circles should not be made on top of one another. The teacher characterizes this first task as “making wholes”: 

22 T: I want you to make them separately on your desk I don’t want you to make them on top of each other. So (I want you to) make as many wholes as you possibly can.

The lesson has begun, then, with concrete action on a tangible and easily identifiable type of object: the plastic pieces. These objects are signified with color terms (e.g., “the blue ones”) and with other terms that index their physical properties: “different shapes,” “pieces.” These segments of colored plastic are objects we can all see – their instantiation is material and self-evident; they are indeed “sensuously, premethodically given” (Lachterman, p. 125). They are acted on by being manipulated, physically arranged, to “make” a given shape (a complete circle).

This done, the teacher orients the students to one of their “wholes,” the one made from a single plastic piece, colored light blue:

56 T: Alright everybody, now I want you to look at the solid piece, which is the whole, the light blue. Is your solid piece.

But she immediately shifts the referent of the term “whole” from this “solid piece” back to all the circles they have made:

62 T: Now, you have the light blue, the whole, right? How many wholes do you have on your desk?

63 S: One

65 T: Six. Six wholes. Each one the color- when you put the fractional parts together, six wholes. This is a whole, whole, whole. [Pointing?]

The student’s answer “One” (line 64) demonstrates the ambiguity in the designation of “whole.” The teacher marks this answer as incorrect, and she provides the acceptable reply (“six”). But “one” would have been correct just a moment ago, when the teacher was referring to “the whole” [emphasis added], the light blue […] the solid piece.” The answer “six” is correct only if “whole” refers not to the blue circle, but any of the circles.

If each of the circles is “a whole,” if there is no longer one “whole” but six of them, then the color and the number of plastic pieces a circle is made from are no longer relevant. As our analysis proceeds we shall see that the teacher progressively eliminates mention of these aspects, which no longer distinguish and contrast the objects under consideration.

### Dividing Each Whole into Components

Classroom talk and activity now center on these “wholes” or “fraction circles” (67). The teacher initiates an extended interaction in which they “start looking at equivalents” (72) of a whole. The teacher consistently refers to this course of action as “dividing” – the students are to divide each whole into its components. The teacher leads the students through a sequence of these “divisions,” first into two pieces,
then three, then four, six, and finally eight. But this is not a matter of simply reversing the previous assembly, because the components into which the wholes are divided turn out not to be the colored pieces, but a new kind of object. The wholes are being divided not into plastic pieces, but “fractional parts” (one might also call them “primitive fractions”). The plastic pieces have color, shape, and a specific number fit together to form a circle. The fractional parts, in contrast, have different properties. Each has a name (“fourth,” “half,” “sixth”… ) that identifies it on the basis not of its immediate material character, but the portion it makes of a whole. These fractional parts can be sequenced: “half,” “third,” “fourth,” “fifth,” “sixth” and so on, while the colored pieces cannot. (And in retrospect it seems that the shift in the teacher’s use of the term “whole” marked a similar shift: the six complete circles must also be considered this new kind of object: not colored circles but fractional wholes (cf. 67)).

As the class “divides” each whole into its component parts these properties become salient. The divisions are carried out in sequence (on line 114 the fact that they have no “fifths” causes some confusion, making it clear that the students are attending to the sequence). The number of parts that make up a whole is identified in each case (e.g., “How many fourths do we have?” “Four” (98-99)), and each fractional part is named. In these respects the fractional parts are truly products of dividing the wholes; they did not exist prior to this activity.

67 T: Okay. So. Let’s divide our fraction circle into halves. Which color
68 Ss: Yellow.
69 T: Yellow, okay, and you have how many pieces?
70 Ss: Two.
71 T: Okay.

The teacher has talked in terms of color and number to guide the students, but she quickly drops these concrete designations:

82 T: So look a the next division.
83 What would you go to?
84 Ss: Three. [Some say “Dark blue,” others say “three”]
85 T: Okay the dark blue, which would be what?
86 Ss: Three.
87 T: Three, and your fractional part would be one third.

The “next” whole, which was made of six dark red plastic pieces, is identified first by its color and then by the number of pieces. The students are apparently iterating from the previous whole, because they claim that this whole has “five” pieces, and is composed of “fifths.” Their procedure is correct but it achieves an incorrect answer: the iteration must jump from “fourths” to “sixths,” and the teacher makes the appropriate correction (121):

111 T: Right. Let’s look at the next one.
112 S: Dark red.
113 T: Okay the dark red. How many pieces do we have?
114 Ss: Five
Learning Math as Ontological Change

115  T:  Five.
116  Ss:  Fifths!
117  T:  So are we talking-
118  Ss:  Dark red.
119  T:  Dark red. Three sixths.
120  S:  It should be one five.
121  T:  Yeah. So it’s gonna be one sixth- you have six pieces right [to S in
122  120], what’s the denominator?
123  Ss:  Six.
124  T:  Six, right. Okay. So one sixth, plus one sixth plus-

With the final division, into eight orange pieces, the students avoid the error of looking for a
circle they can divide into “sevenths,” and don’t even need the color term:
130  T:  And the last one is?
131  Ss:  Eight.
132  T:  Eight! Okay. But we’re gonna do the same thing.

They are apparently becoming skilled in the dividing in which colored circles are ontologically
transformed into fractional parts.

“Whole” becomes a fraction

Note that the referent of “whole” has shifted again. When the teacher says “but it’s still a whole,
 isn’t it?” (line 78) her formulation implies that something is unchanged when the yellow fraction circle
has been divided into its parts. The term “whole” is now being used to refer to something that retains its
identity even when it has been disassembled (“divided”) into its components:
78  T:  Okay we divided in a half, okay. We have two pieces here, but it’s
79  still a whole isn’t it?

We saw that at first the term “whole” was used to refer to the light-blue, solid circle. At this point it
would have been odd to say that this was “the same” as two yellow semi-circles. Next, “whole” referred
to each of the assembled circles, no matter its color or the number or shape of its parts. But still here if, for
example, the two yellow semi-circles were separated there would have been only five of these “wholes,”
no longer six. Now the term “whole” is being used to refer to a species of mathematical object that retains
its identity when it is divided into its component parts. What kind of object can this be? We propose that
this new kind of object is the fraction. To trace this ontological change and this new type of object, we
need to consider what the teacher has been writing.

Symbolic Reproduction

We have deliberately left something important out of our account of the lesson thus far. While
the enactive division of wholes into component fractional parts is taking place, the teacher is putting
these pieces together again, reproducing the whole, symbolically, by writing on the chalk board. We
suggest that this inscription is of central importance in the transformation of the wholes and (combinations of) fractional parts into fractions. Let’s examine the details:

Once the students agree that the first whole is the yellow one, and that it is made of two pieces, the teacher says, “Okay, so I’m gonna do something, and I want you to start looking at equivalents” (72). She writes “1” on the board, and continues, “Okay we have a whole, one, okay? And then we divided the next one we had yellow, okay, right? Okay we divided in a half, okay? We have two pieces here, but it’s still a whole isn’t it?” (75-79). The students concur.

The teacher says, “Let me do it like this it’ll be a little easier” (81). She writes:

\[ 1/2 + 1/2 = 1 \]

and says “one half plus one half equals one, okay” (82).

When the class moves on to the whole made with blue pieces, the teacher says, “So it’d be, one third, plus one third, plus one third, equals?” (87), while writing:

\[ 1/3 + 1/3 + 1/3 = \]

But before completing the equation she undertakes a self-initiated repair. She goes back to modify the previous formula with halves, saying as she does so “Let me do it like this so you can see the difference” (89). She inserts the symbol “2/2.” Thus:

\[ 1/2 + 1/2 = 2/2 = 1 \]

...as she says, “Would be- would equal two over two which would equal one” (90). Then she returns to the case of thirds. “So one third plus one third plus one third would equal what? Three over three, equals one, okay. Get it now?” (90-94). And she completes the equation:

\[ 1/3 + 1/3 + 1/3 = 3/3 = 1 \]

What is accomplished here? We propose that the teacher is demonstrating that the fractional parts “thirds” are not only the objects into which a whole (blue) “fraction circle” can be divided, they also can be used to make yet another kind of mathematical object: the fraction that is “one,” and that is also “2/2” and “3/3.” On this interpretation, when the teacher rewrites the equation so the students “can see the difference” (89), what she’s speaking of is the difference between two species of object: fractional parts, and fractions. This same difference would also be the “it” when she asks “Get it now?”

**Plastic Pieces, Fractional Parts, and Fractions**

Let’s consider the properties of fractions, as we have considered those of fractional parts and colored pieces. Fractions (such as 3/6) can be considered combinations of fractional parts (primitive fractions). They are not components of a whole, since a specific number of the same fraction cannot always be combined to form a whole (e.g. several 3/8s cannot be summed to a whole). Nor can they easily be sequenced. Their names are not simple but compound (e.g., ‘three eighths,’ ‘three over eight’). And importantly, while two primitive fractions can never be identical or equal, two fractions, such as 3/4 and 6/8, can be ‘equivalent.’

On this interpretation, when the teacher writes a formula such as:
Learning Math as Ontological Change

1/2 + 1/2 = 2/2 = 1

she is inscribing both the difference between and the equivalence among two types of mathematical object: fractional parts (written as ‘1/2’ and spoken as ‘one half’) and fractions (written as ‘1’ and ‘2/2,’ spoken as ‘two over two’). Fractions, then, can be instantiated as arrangements of fractional parts (e.g., three contiguous sixths – which themselves are instantiated as red pieces – forming a segment) and also as inscriptions on the chalkboard: ‘3/6.’

It is among these objects – fractions – that the relationship of “equivalent” is now sought. Equivalents are both verbally stated – “six over six equals one”; “one half plus one half equals one” – and inscribed: “1/2 + 1/2 = 1”; “1/2 + 1/2 = 2/2 = 1.”

It is important to notice that the inscriptions retain traces of the fractions’ production. The inscription 2/3 reminds us that the fraction has been made by combining two fraction parts, each of which was one “third.”

There may appear to be ambiguity to the verbal and written designations. For example, the inscription “1/3” is ambiguous: it could designate a fractional part or a fraction “Three sixths,” for instance, could refer to an arrangements of three fractional parts (three ‘sixths’), or to a single fraction (‘three-sixths’). The same is true for the inscription “3/6.” But the ambiguity can generally be resolved by attention to context, either verbal or non-verbal or both. It is helpful to notice that fractional parts can be counted over, while fractions cannot (we can count the number of “thirds,” but not the number of “two thirds”). And we do indeed find that fractions are typically referred to in the singular (“two thirds is”) when fractional parts would be plural (“two third are”). For example, “When we’re talking equivalents is [sic] two halves equal to one whole?” (line 141).

It is also important to appreciate that the transformation of plastic pieces into fractional parts and then fractions is not merely a matter of a single referent receiving a new reference (i.e. the same object just given a new name). The referent is not the same, because the new terms of reference invoke a different field of objects, of which the referent is just one member. So, reference to ‘a blue one’ invokes a field made up of ‘a green one,’ ‘a red one,’ and so on, while reference to ‘a half’ invokes a different field: ‘a quarter,’ ‘a third,’ etcetera. Reference to a fraction such as ‘two quarters’ invokes yet other fields, such as that inhabited by ‘four eighths,’ ‘six twelfths,’ and so on (all of which are in fact the ‘same’ fraction). Correct identification of the object referred to thus requires correct recognition of the context in which it is being projected.

This process rests crucially on properties of natural language, such as its indexicality, contextuality, and historicity, that are absent from the formal expressions of mathematics. We see an illustration of Rotman’s point that math is sustained by informal practices. While formal mathematical expressions such as the equations the teacher is writing on the board are an important product of the activity, the informal discourse between teacher and student is a crucial support for these inscriptions.

The Power of Inscription
The importance of written symbols in mathematics has often been highlighted. Rotman writes, “being thought in mathematics always comes woven into and inseparable from being written.... Thinking in mathematics is always through, by means of, in relation to the manipulation of inscriptions. Mathematics is at the same time a play of imagination and a discourse of written symbols” (1993, p. x, original emphasis). And Sfard and McClaim (2002) have a argued that as the term ‘representation’ has been replaced by ‘inscription’ there has been a shift from viewing symbols merely as ‘external’ representations of inner mental constructs, to seeing them as tools, cultural artifacts, which mediate activity and relations, and are intrinsic components of mathematical goings-on, a shift also from attending to the state of children’s knowledge to a focus on actions of knowing (Sfard & McClain, 2002, p. 155). Cobb (2002) uses the term “chain of signification” “to illustrate a way of accounting for mathematical learning in semiotic terms” (p. 191). But Cobb does not explore the power of chains of signification to change the subjectivity of the math learner.

What is the role of inscription in this episode? First, of course, inscription introduces the conventional representation of fractions, as one number, the ‘numerator,’ ‘over’ another, the ‘denominator.’ In this way it codifies the part/whole relation central to fractionality – ‘fraction’ is literally the act of breaking – and thus supports new ways of talking. The phrase ‘two over two’ is literally a description of what has been written on the board (‘2/2’). The terms ‘numerator’ and ‘denominator’ – as when the teacher says “the denominator’s gonna be the bottom number” (154) – refer to the structure of the written symbolism for a fraction (cf? 358, 362, 386.)

Second, the inscriptions show the relationship between the fractional parts and the fraction. As we described above, the equation “1/2 + 1/2 = 2/2 = 1” shows how fractional parts can be combined to form (“equal”) a new and different kind of object. Such equations state both the ontological difference between types of object – fractional parts and fractions – and the equivalence of fractions. Writing facilitates the ontological changes we are pointing out because inscription serves to make visible relationships between these objects, the fractional parts and fractions, that would otherwise be hard to see. As Rotman (1993, p. 143) notes, “what is essential about [mathematical] signifieds is their intersubjective control and regulation by written inscriptions.” The written equation represents new imagined objects and imaginary actions upon them.

At the same time, third, once several of these “dividings” have been completed, the inscriptions make evident the relationships among these different actions. For example, the two equations “1/2 + 1/2 = 2/2 = 1” and “1/3 + 1/3 + 1/3 = 3/3 = 1” represent and make visible two successive (and sequential) actions as simultaneous inscriptions. The relationship between these two “dividings” can now be examined. In this manner the inscriptions begin to reveal the logic of a system: an assemblage of signs operating as a whole; an organized set of procedures. In short, a discourse: signs that run on in an orderly manner.

Fourth, the inscriptions become objects in their own right: for example, the symbols can be counted, as when the teacher counts the number of times “1/6” has been written on the board before
School Math and Ontological Change

Fifth, inscription provides a way of remembering the actions the students have carried out. While activity with the plastic pieces is fleeting, leaving as a record only its product, the inscription is enduring: it remains on the chalkboard for all to see. By producing visible traces, inscription provides a historical record of the activity of “dividing” the wholes. Rather than the students needing to remember what they have done, the inscriptions do the remembering for them, in an autonomous and automatic way. The teacher will exploit this record shortly.

Recontextualization

As wholes are divided into thirds, fourths, sixths, and finally eighths, and as corresponding inscriptions are produced, changes are evident in the contribution of teacher and students. Over the course of the lesson the teacher works to transform the students’ participation. The dividing of one whole was accomplished mainly by the teacher, while the “making” of equivalents to one half was mainly the students’ accomplishment. Similarly, we saw that when the whole was divided into thirds the teacher explicitly named the color and the fractional part, formulated the equation, and stated the fraction and its equivalence to a whole:

... So look at the next division.

What would you go to?

Three. [Some say “Dark blue,” others say “three”]

Okay the dark blue, which would be what?

Three.

Three, and your fractional part would be one third. So it’d be, one third, plus one third, plus one third, equals-? [1/3 + 1/3 + 1/3 =] This is what I should do. Let me do it like this so you can see the difference. Would be- would equal two over two which would equal one. [2/2 = 1] So one third plus one third plus one third would equal what?

Three over...

Three over three, equals one, okay. [3/3 = 1] Get it now?

(As)

Oh wonderful! Okay, what’s the next fraction?

But when they divided the whole into fourths the teacher omitted any mention of color and formulated the equation by eliciting its elements from the students:
By the time they reach division into eighths, the students are able to identify the number of fractional parts (“Eight”), to formulate the equation orally (chanting “One eighth plus one eighth plus one eighth...” the correct number of times), and to complete the teacher’s formulation of the fraction and its equivalence:

130 T: And the last one is?
131 Ss: Eight.
132 T: Eight! Okay. But we’re gonna do the same thing. One eighth plus
133 one eighth plus-
134 Ss: [Ss join in then take over, but chant, some eight times.] [T laughs] [Ss laugh]
135 T: Sound like a bunch of robots! One two three four five six- okay, plus
136 one eight plus one eighth. [T writes: \[1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8\] Do I have eight of them up here?
137 Ss: yes.
138 T: [Counts the items on the board] So it’ll be eight-
139 Ss: -over eight equals one whole.

This “robotic” chanting bears closer examination. The children speak as though with one voice, as though the chain of signifiers unfolds automatically, as though it is telling them what to say. The “chain of signification” has become a chain indeed: its links binding not only its terms but the children too, constraining what they will say. They give voice to the knowledge that is locked into the links between signs. We will argue below that this is strong evidence that a new form of subjectivity is being formed, one suitable for infinite iteration.

From Diving into Fractional Parts to Making Fractions

Once all the fraction circles have been divided the teacher leads the students through a reiteration of what they’ve found – the different equivalents to one whole – a reiteration made possible now that each dividing is inscribed on the board:

141 T: Now. When we’re talking equivalents is two
142 halfs equal to one whole?
143 Ss: No; yes.
144 T: Okay. What else is equal to one whole?
145 Ss: Three thirds.
146 T: Three thirds.
At this point the teacher marks a transition:

Okay, six sixths. Okay. Now. Now what I want- ask you to take these and I want- we’re gonna deal with- you- you gonna use- choose the right color. ( ) I’m gonna give you a fraction, the denominator’s gonna be the bottom number, that’s the color you should look at. So if I say to you, take one half and place it on your paper, (to) take one half and place it on your paper, what color would you look at ( )?

The students begin a task that seemingly differs from what they have just done only in that they are working with “one half” instead of “a whole.” But finding the equivalents to a half, unlike finding those to a whole, is cast as an action of making rather than dividing. For example: “[T]ake two from the white circle and make a half” (167, and cf. 162, 178, 243). The students are now “making” a target fraction from its fractional parts. In other words, the students are now doing orally what the teacher was previously doing with inscriptions. Previously the students had been ‘dividing’ each whole into fractional parts, while the teacher had been combining these parts (on the chalkboard) to make a fraction. Now the students are making the fraction.

And so it is appropriate that although the teacher continues to write on the board, she now inscribes only the product of the task (e.g. “2/4,” “3/6,” “4/8,” “5/10”). She doesn’t bother to inscribe either the target fraction or the parts used to make it, only “the equivalent,” the “fraction.” Notice that this was precisely the element she earlier added to her inscriptions when the students were “dividing” the whole. It seems that the fractional parts, now ready-to-hand, can be put together, and the fraction “made,” in thought without the support provided by the former being inscribed.

A further difference is that the activity is no longer tied to “wholes.” The process that has been established for finding the equivalents to a whole is being generalized, and iterated. Here one can see, we suggest, the order, the seriation, to method that Lachterman and Rotman describe as central to modern Cartesian mathematics.

As though to support the students in this shift to a new degree of abstraction and the loss of the anchor of the whole [FN], the teacher returns momentarily to the concrete properties of color and shape. But she quickly shifts back to “fractional part” terms when a student makes a tentative formation (164):

The quickest color, the quickest one?

Yellow.

Yellow. Okay. What else could you take that would make- uh, from the white that would equal one half?
164  Ss:  two-
165  T:  Two what? Two fourths? Very good. From, the dark blue.
166  Ss:  None?
167  T:  Right. Look at- take uh, one half and put it on the white sheet, take
168  two from the white circle and make a half, now take two from the
169  the dark blue. Put it on your circle. Is the dark blue the same
170  size as the yellow and white? [said with emphasis]
171  Ss:  No.
172  T:  No. Is that a equivalent?
173  Ss:  No.
174  T:  No. So you know that two thirds will not be the same or equal to two fourths or?
175  Ss:  One half.

Of course there are several ways to “make” “one half.” The class moves sequentially through two
fourths, four eighths, and three sixths. Then the teacher again calls for a verbal reiteration of the
“equivalents” they have identified:
207  T:  What are your equivalents of one half? Okay, start with- the white,
208  this is the way we want to-
209  S:  Two quarters.
210  T:  Okay two fourths. [T writes] What else?
211  S:  Three sixths.
212  T:  Three sixths [T writes]
213  S:  Four eighths.
214  T:  Four eighths. [T writes] So if I should ask you Joel, give me an
215  equivalent of one half, what would you tell me?
216  S:  Two fourths, three sixths, and four-

Three sixths: clue on the board
A further iteration of the activity is begun as the teacher asks the students to “give me the
equivalents for” one third (230-287). This proceeds in much the same way as the previous iterations. The
teacher next asks for “the equivalent of three fourths” (290), but then – with another change in
participation on the part of the students – she asks one of them to pick the next target fraction.

The selected student picks “three-sixths” (337). The class has already found equivalents to one
half, which of course would give the same results, and another significance of inscription becomes
apparent here. The teacher tells them (346), “I see a clooo on the board,” gesturing towards the
inscription: “1/2 = 2/4 = 3/6 = 4/8.”
344  T:  I see a clue on the board. [There’s already written: 1/2 = 2/4 = 3/6 = 4/8]
345  Ss:  Ooo!
346  T:  I see a clooo on the board!
347  Ss:  Cloo!
Learning Math as Ontological Change

348  T:  [soto voce] I see a clue on the board.
349  S:  She gave us a easy one.

XXX. The teacher finally says,
373  T:  Excuse me- you (don’t mind-) I tell you all the time there are clues?
374  T:  Excuse me, stop. Clue. [She points at the fractions on the board and
375  waits. Silence for several seconds, then students start to laugh.]
376  Ss:  Yes, yes yes!
377  T:  Okay. A lot of times you work hard when you don’t have to.

The record of past action that inscription provides a resource for avoiding unnecessary repetition of work.

Inscription occurs a final time in this task (287) when the class finds the equivalent to one third. After this, equivalents are found without the aid of inscription.

Iteration Beyond what is Given

We have traced how this math lesson iterates through a sequence of target fractions: a whole, a half, then a third, before the teacher makes the target more complex, three fourths. For each of these targets, the activity moves through another level of iteration, as the students find equivalents in sequence: for example, a whole is two halves, three thirds, four fourths, six sixths, and eight eighths. Rotman finds at the heart of Cartesian math the supposed capacity for infinite iteration – the *ad infinitum* principle.

Lachterman writes of modernity’s “projection of *ad infinitum*” (p. 3), and cites Blais Pascal: “Man is produced only for the sake of infinity” (Pascal, 1954). It is highly suggestive, then, that the lesson concludes when the students demonstrate that they can iterate beyond the fractions that are instantiated in the plastic pieces on their desks. They have been finding equivalents to three sixths. After pointing out the “clue,” the teacher asks rhetorically whether one half, and the equivalents to one half – two fourths, four eighths – are not also equivalents to three sixths:

378  T:  Now when you put all those together, you put three sixths
379  Ss:  there, if you put one half doesn’t that equal three sixths?
380  Ss:  Yes.
381  T:  Isn’t that the same size? Well if you put two fourths, ‘scuse me, isn’t
382  that the same size as three sixths?
383  Ss:  Yes.
384  T:  If you put four eighths isn’t that the same size as three sixths?
385  Ss:  Yeah.

Then she declares, “so is six twelfths but we don’t have twelfths”:

386  T:  Yes. Yes, and so is six twelfths but we don’t have twelfths. We don’t
387  have the denominator for twelve, we don’t have the denominator- what
388  else would it be, sixth twelfths, what else?
389  Ss:  Eh-
390  T:  What else could you go up to?

11/6/14
Learning Math as Ontological Change

391  S:  Twelve twenty-fourths.
392  T:  Twelve twenty-fourths.
393  S:  Sixteen thirty-second- twos.
394  T:  Thank you!  [Laughs] Alright. [She has them put the plastic pieces away.]

When she asks them, “What else could you go up to?” and the students (at least some of them) reply “Twelve twenty-fourths,” and then “Sixteen thirty-second- twos,” they display a capability to “go up to” fractional equivalents that they don’t actually “have” – they demonstrate an ability to iterate beyond what is immediately given. That’s to say, the students here perform a kind of action that is possible only on the new type of object that has been created. The relationship of equivalence that holds between fractions is such that, given one fraction, the next in a sequence can be identified, indefinitely. This isn’t the case for the plastic pieces (given “orange” one cannot “go up” to some other color), nor for the fractional parts (given “one eighth” one could move ambiguously to “two eighths,” or “one sixteenth,” or “one ninth”). So the students’ ability to iterate from four eighths to six twelfths, twelve twenty-fourths, and sixteen thirty-seCONDS demonstrates – to the teacher and also to us – that they correctly recognize the species of object on which they are acting. That the teacher also recognizes this is strongly suggested by the fact that she ends the lesson at this point.

Learning Mathematics as the Production of Objects and Subjects

We have seen that this mathematics lesson does indeed involve an ontological level, not only in the sense that different mathematical objects are addressed, it also involves the production of new species of mathematical objects. The lesson is such that “the objects of the discourse” that “eventually stand out from the intricate web of utterances” (as Sfard put it) are three distinct kinds of mathematical object, of increasing abstractness. First are the colored plastic pieces – “the light blue one.” From these the students “make” wholes, which they then “divide” into a second kind of object, component “fractional parts”: “halves,” “thirds,” and so on. From these fractional parts yet another kind of object is made, “fractions” such as “two sixths,” “four eighths,” “four fourths.” It is among these objects, the fractions, that “equivalences” are identified. We have noted the iteration central to this activity of finding equivalents, and the indefinite iteration – ad infinitum – with which it ends.

These distinct types of object are talked about differently, acted upon differently, and are dealt with differently in inscription on the chalkboard. And they form an ontological progression: fractional parts are based upon plastic pieces; fractions are based upon fractional parts. A “whole,” for example, starts life as “the light blue” but is transformed first into any complete combination of “the fractional parts,” and then into the fraction written variously as “1,” “2/2,” “3/3” etc., and spoken as “one,” “two halves,” “three thirds,” etc.

Evidence of this ontological progression is provided by occasions when the teacher corrects the students’ misrecognition of the appropriate objects. We’ve already seen the shift in signification that occurs in lines 62-65, when a student mistakenly identifies “whole” with “the blue one.” As the lesson progresses, the students’ use of color terms becomes at times illegitimate in the discourse. When a student, struggling to come up with an answer to the teacher’s request for “an equivalent of three sixths”,

11/6/14
answers, “Three of the reds” (355). The teacher responds, “But I want the fractions. I don’t want the colors now” (356), thereby indicating that the student has not indexed the proper object. The student has identified the appropriate piece of plastic – the sixths are indeed red – but not the appropriate object: a fraction, not a plastic piece. Transformation in the objects acted on can also be seen in changes in the way the teacher describes the tasks. While in line 176 she asks a student to pick “another circle,” by line 331 the request is to “give us a fraction.”

**Production of Objects**

How are these mathematical objects created? What kind of process are we seeing here? What does it mean to say that discourse creates objects? Sfard says of philosophical discourse something that is equally true of everyday classroom conversation: “we always imply the existence of another entity for which the entity at hand (a mark on paper, a word) is intended to be a discursive replacement” (19XX, p. 15). That’s to say, the signs and symbols, verbal and written, of classroom math imply the existence of other entities (cf. conversational implicature, Levinson, 1983, p. 97ff).

Hanks (1996) suggests that “the referential process is one in which subjects, objects, and social relations are simultaneously produced in the course of even the most mundane utterances” (p. 237). Hanks draws from Ingarden (1973) the notion of a text’s “intentional objects” and Merleau-Ponty to “the agent’s ability to project a world of new objects in every act of speech” (p. 136),

And Shweder (1991) explains how “[a] sociocultural environment is an intentional world. It is an intentional world because its existence is real, factual, and forceful, but only so long as there exists a community of persons whose beliefs, desires, emotions, purposes... are directed at, and thereby influenced by, it” (p. 74). He considers how “intentional (made, bred, fashioned, fabricated, invented, designated, constituted) things exist only in intentional worlds” (p. 74).

The intentional world of this classroom – ethos and ontology – is evidently that of modern, Cartesian mathematics. Fractions were of course known to classical Greek mathematicians, and this lesson began with – and when necessary returns to – some classical techniques. (The use of segments of the circle grounds this lesson in geometry, though it ends in algebra. The use of practical and sensory techniques such a superimposing segments of different colors on top of one another to see what “fits” is an appeal to congruence, a technique which even Euclid found dated and questionable.) But, unsurprisingly, there is much about this math lesson that is Cartesian. Its “dividing” and “making” clearly exemplify the “construction” of objects which Lachterman identified as central to the ontology and ethos of Cartesian mathematics. Each student works with his or her own materials, and in this respect the “dividing” and “making” appear to be individual actions, and are described as such. When the teacher asks questions of the second-person plural, the implication is that each student will have the same answer because the task is an objective one that permits only one correct solution. But in fact this individual activity is coordinated as a whole-class interaction, and the objects on which the students act are produced by collective participation in social practices. The task is an intersubjective one, resting on shared fictions, its common solution requiring the willingness and ability to dream together.
Abstraction

Children’s development is often conceived of as a movement from dealing practically with particular objects, immediately present, to dealing intellectually, mentally, with abstractions. Development is conceptualized as a movement from concrete particulars to general abstractions. But this math lesson illustrates that the relationship between abstractions and concrete objects is more complex than this: abstractions render concrete objects recognizable (cf. Egan, 19XX, p. 47). Perhaps it is better to think of the classroom activity as a game which shifts so that while at first playing it requires manipulation of plastic pieces, along with words, eventually the play is purely verbal. Along the way the vocabulary shifts so that the objects in the game are either plastic pieces or representations (written on the board, or spoken). “A red one” can refer only to a plastic piece, while “a quarter” can refer to a plastic piece, or a combination of several plastic pieces, or writing on the board, or other words (such as “two eighths”).

The move could be said to be from actual objects to fictive ones, from real to virtual, from actual to imagined actions on imaginary objects. But this imaginary action is not the product of a logically necessary adaptation, it is a socially motivated and intersubjectively sustained “web of waking dreams, conscious fantasies” (Rotman, 1993, p. 86). Rather than seeing it as indicating a level of cognition that is more advanced, more systematic, more adequate to reality, we must ask about the “forgetting” that is required, the denial of time and place, …

Production of Subjects

Thus far we have focused on the production of mathematical objects. But the corollary to this is the production of mathematical subjects; production of the kind of people who can engage in mathematical activity. We are not claiming that these 5th graders have become mathematicians, but there is evidence that they have changed. Mathematics, Rotman proposes, is “a kind of fantasy action” (p. 139), “its entire discourse consisting of a web of waking dreams, conscious fantasies” that operate on a separation between activity performed or performable by the Subject and that which is dreamed, imagined to be performed by the Agent” (p. 87). The fantasy, as Valerie Walkerdine has pointed out, is “the dream of a possibility of perfect control in a perfectly rational and ordered universe” (Walkerdine, 1988, p. 187). Rotman offers a model of the psychological consequences of mathematical reasoning, understood as “chains of imagined actions that detail the step-by-step realization of a certain kind of symbolically instituted, mentally experienced narrative” (p. 66). Mathematical texts, and lectures, “in their instructions to manipulate mathematical signs,” “invoke” specific “agencies” (p. 68). The formal mathematical code appeals to a “common, idealized sign-using agency” which Rotman calls “the mathematical Subject” (p. 70), while the informal, everyday social and historical practices of mathematics appeal to the “Person.” The Subject is an “idealization,” XX. “To understand the relation between these agencies as well as their joint contribution to the figure ordinarily and uncomplicatingly called the ‘mathematician,’ we need to look first at the grammar of mathematical texts” (p. 71), which are filled with imperatives, and they lack indexicals. The Subject must respond to inclusive imperatives (“Let us define…”), but exclusive imperatives (“Construct a
triangle...") cannot be carried out by the Subject itself, for they go beyond its capabilities; it needs a proxy, XX “imagines into being an idealized simulacrum of itself as its surrogate, what C. S. Peirce calls a “skeleton diagram of the self,” which executes exclusive imperatives of the form ‘count,’ ‘integrate,’ and so on” as imagined actions in an imagined world. This is the “Agent,” an “automaton” which operates without semantics only on “signifiers at a sub-Coded level.” The imago of the Agent has to be “something transcendental,” “a wholly disembodied, immaterial phantom” in order to “count endlessly for us.” It is “a business of manipulating inscriptions that characterizes mathematical thought as a kind of waking dream” (p. xii).

The Subject has available only the mathematical code, and since this lacks any indexicality, the Subject is denied access to any semiotic resources to describe its own subjectivity. “This denial manufactures the Subject as an a-historical, a-cultural, a-social truncation of the Person, a nameless trans-individual operating under a universal psychology” (p. 74). In particular, the Subject must assume, but cannot articulate, resemblance between itself and the Agent.

The structure is analogous to that of a dream – “the Agent maps onto the figure dreamed about, the Subject the dreamer dreaming the dream, and the Person the dreamer awake” (p. 78) – but mathematics is a “waking dream.” Any adequate history of mathematics must include “the purely formal problems and symbolic developments of the Code..., the cultural institution and legitimation of the Agent, and... the desiderata and demands of instrumentality made... by science and technology” (p. 82).

Equally, any adequate diachronic account of learning mathematics must include the formal statements, both inscribed and spoken, the cultural legitimation of specific imagined worlds and agencies, and the informal practices and relations that sustain and motivate the enterprise. Our analysis of classroom discourse attends to the first and second of these three “the constitution and ontological status of [mathematical] objects is inseparable from those cultural and historical practices” of “motivation, legitimation, transmission, institutionalization, and implementation of mathematics” (p. 82).

This semiotic model - which attends to the way to do math is to manipulate mathematical symbols and think mathematical thoughts - “depicts mathematical reasoning as consisting of certain imaginings or thought experiments” on several levels. The “mathematical Subject” is “a semiotic agency made available by the [mathematical] Code” (Rotman, 1993, p. 145).

The lesson ends, as we have seen, with the students iterating one last time, and this time the iteration is not bounded by the sensuous properties of those objects—the plastic pieces—that are immediately present. The students, at least some of them, have demonstrated an ability to iterate seemingly indefinitely... and this is possible only because the objects now being named in the iteration are no longer colored pieces, but fractions. No longer material objects, but ideal objects. When the students demonstrate their new ability to iterate indefinitely, when they demonstrate that they have “got it,” they show they’ve stepped into the fantasy of universal mastery which lies at the core of the ethos of modern math. In Rotman’s terms, through the course of this activity these children have become, albeit perhaps...
only for the moment, “demonstratibly infinite iterators.” As such, we must infer that each has become a “mathematical Subject.”

The difference between ancient and modern mathematics was “not one of competing theories or conceptions of the ‘mind,’ as though this term named a philosophically neutral agency with ancient and modern renditions” (Lachterman, p. 4). The modern construal of mind as actively constructing has to be understood as part of a program of production and productivity that continues today (cf. Baudrillard, 1975). Learning mathematics, then, must be understood not simply as the training of a preexisting entity, but the formation (the production!) of a modern way of being. The fabrication of modern minds is essential for perpetuating “the idea of modernity” (Lachterman, p. 4), and the continued circulation of mathematical signifieds is essential too, for “what we call the ‘world,’ whether this is the techno-scientific theorized world of material process or the commercial world of circulating money and its instruments, is in a vast and continuing debt to mathematics” (Rotman, 1993, 141).

“The counting agency, the Agent who brings the iterates into being, does so as an automaton, a robot diagram of the mathematical subject” (1993, p. 99). cf pp. 103ff

**Academic Tasks as Cultural Tasks**

One of the contradictions or paradoxes of the classroom community is that in and through dependence at one level of activity (that of student and teacher) independence is fostered at a new level (that of mathematical activity). When the children come to operate on fractions, to iterate indefinitely on objects they “don’t have” concretely, they are entering into this fantasy of power and mastery – by following the directions of an authority.

Lave (1988) has pointed out that while street math is typically an end in itself, school math is generally a means to an end, and the end is pleasing the teacher. Lave is correct, we think, in seeing school’s academic tasks as means to an end. It is no accident that children in school do math in some sense for, and certainly in relationship with, an adult teacher; indeed, we would argue that this is crucial to the cultural task of schooling. The relational and cultural character of school defines the kinds of problem-solving, skill-acquisition, and intellectual inquiry that occur, and often makes school the site of a search, sometimes a struggle, for identity. When culture and relationships are ignored we can’t adequately understand either schooling’s cognitive or its social aspects, or grasp how schools transform children into adults who will live and work in our complex society. This is not to say that school should not be a target of critique, but criticism should focus on the ends of schooling’s cultural tasks, not simply those of its academic tasks.

The lesson we’ve just examined occurred the day following a field trip to the middle school. This had been a visit before, during, and after which the teacher emphasized the new and difficult challenges her students would face when they graduated to the middle school the following year. From the first day of class she had presented herself as knowing what was important for the children – better, in fact, than they knew themselves. Specifically, she knew how they would need to behave if they were to survive in
Learning Math as Ontological Change

the rough new world of middle school, which we can attest was a large and complex institution, emphasizing discipline and control.

In this lesson it seems that the children (at these those whose voices we can hear in our audio-recording) have accepted the legitimacy of their teacher’s authority; they have accepted her message that to thrive in the middle school they must do what they are told. At the same time that they have become masters of the universe, mathematical “Subjects” capable of, for example, the unbounded activity of iteration, they have also become subjected. This double-edged transformation is, we believe, a central part of what contemporary schooling is about.

Conclusions

We have sought to illustrate that it is possible to describe teaching as a transformative process (or praxis) in concrete terms, using publically-available evidence, and without invoking a ‘black-box’ (student ‘cognition’) that cannot be opened. (The ability to perform mathematical actions ‘in the head’ requires, we would suggest, following Harré in press, a brain but not a mind, insofar as to perform ‘mental arithmetic,’ for example, we do not ‘internalize’ actions but instead willfully exploit properties of our neurological apparatus.)

We have suggested that this task teaches the students math (school math, at least) by socializing them into activity that systematically forms new kinds of object. Over the course of the task they come to both talk about and appropriately manipulate fractions: objects they “don’t have” in the way they have the plastic pieces. At the end of the lesson they demonstrate that they can iterate indefinitely with these new objects.

We have proposed that learning mathematics is a matter of recognizing new mathematical objects, and acting appropriately on these objects. We have shown how the students in this mathematics lesson come to iterate on fractions. Fractions are the mathematical objects which become recognized [perceived?]; iteration is the action performed upon these objects.

In a sense, we have described the results of viewing an academic task not simply as having a social component that as it were accompanies its cognitive demands, but as being entirely social. That is to say, we’ve described an occasion of learning as a matter of changing involvement in social practices. Involvements rests on participation in the resources and constraints of everyday conversation (as well as the math register(s)?) on the prior establishment of a particular kind of relationship between adult and children – that’s to say, as teacher and students. And on grasping the indexical relations between talk and world.

The notion that learning mathematics is a matter of socialization is a useful one. But this socialization is not simply a matter of learning the “ways of knowing” of a new community. Learning school math certainly involves legitimate participation in the discursive practices of the classroom, and the institutional order of the school. But these practices bring about not just epistemological but ontological change; not just new ways of knowing but new ways of being. These ontological changes include the formation of new kinds of mathematical object: “fractions,” ideal objects that can be acted
upon even though they are not immediately present. And as the children come to participate in the practices that form these objects they too, we have suggested, are ontologically changed. What happens in the classroom is not just mathematical ways of knowing but entry into a world of mathematical objects, a world within which the child becomes a different kind of “Subject.”

Modern math rests on the fantasy of mastery, and offers the dream of a rational, controllable universe to its practitioners. In return for being orderly, for following orders, for putting and keeping things in order, the children are offered the power to generate order, and to extrapolate it without end. These students’ willingness to align with their teacher, to work for her and seek her recognition, is both the basis for and the product of the Faustian bargain in which an academic task achieves cultural ends. The children are not just learning math, they are learning to be a particular kind of person.

We conclude with the suggestion that math instruction should be considered against the backdrop of the broader agenda of schooling. Debate about what constitutes effective math instruction, for example, should make reference not just to the math competencies and knowledge we wish children to have, but also to the kinds of adult we want a particular group of children to become.
We turn to Rotman’s work throughout this paper because he is one of the few people to systematically explore this new view of mathematics. Rotman’s work is not very well known by developmental psychologists or educational researchers; better known is the writing of Valerie Walkerdine (1988), who Rotman has influenced.

For Rotman, “Mathematics is before all else self-consciously produced; and it is so according to an agenda formed out of its historically conditioned role: as instrument in relation to the needs of both commerce and technoscience and, with greater autonomy, out of the image of itself as the exercise and play of pure, abstract reason engaged in the production of indubitable truths” (Rotman, 1993, p. 25).

Because the term ‘construction’ is commonly associated with both Piaget’s genetic epistemology and social constructivism (most of which is solely epistemological in its analysis) we will generally use the term ‘production’ instead.

Seeger (1998) describes a study by Clemens and Del Campo which shows that children do not spontaneously or naturally perceive parts of a circle to be fractions. Seeger concludes that “even very simple and basic representations of fractions have to be learned” (317).

Riddle & Rodzwell propose that children’s prior, informal knowledge of what makes “a whole” is a good place to begin formal instruction. “Most of our students obviously understood that fractions are parts of wholes. They used that understanding to create successful procedures for adding. They pulled off existing wholes, patched together pieces from the remaining fractional portions, and put all the wholes together. As she analyzed her students’ work, Maryann Young, a third- and fourth-grade teacher, remarked, ‘It’s that ‘wholes thing’ again. They keep trying to fill up the wholes’” (Riddle & Rodzwell, 19XX).

Webster’s defines “iterate” as: “to say or do again or again and again.” Cf. line 132: “But we’re gonna do the same thing.”

It is not insignificant, we’ll see later, that the teacher refers to the first of these targets as “our fraction circle” (67), and to each subsequent target as a “fraction” (154, 229, 240). (Line 96 is a rare exception to this pattern.)

The term “equivalent” is used three times in the teacher’s preamble to the task, then not again until line 141 when the first reiteration begins: “Now. When we’re talking equivalents is two halves equal to one whole?” (The term “equivalence” is never used.)
References


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