# Assessing the scientific concept of number in primary school children*

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ABSTRACT

Davydov's mathematics curriculum is aimed at developing a scientific (theoretical) concept of number starting in Grade 1 (age 6–7 years). It is based on a radical restructuring of both the content and the form of classroom learning. The question arises of how to assess the level of development of the concept of number in such young children. Some results are presented of three different assessments of the concept of (whole) number in Grade 1 and Grade 2 students in public schools in Portland, Maine, U. S. A. The assessments take the form of: (1) object-oriented tasks, (2) brief written assessments, and (3) ongoing work with custom-made computer programs. Correlating these three assessments suggests that in the given school population, it was not possible to develop a scientific concept of number in Grade 1, but that this was possible by the end of Grade 2. Complete success (close to 100% of the students demonstrating the concept of number) was achieved only in the classroom where Davydov's form of instruction was fully accepted by the students. The mathematical assessments were complemented by an evaluation of the children's level of development of the analysis, planning and reflective levels of theoretical thinking, based on the use of a test created by A. Z. Zak.

## I. INTRODUCTION

Mathematics instruction in the first years of primary school continues to be problematic for teachers. Some children are able to orient themselves in the mathematics curriculum with ease from the beginning of Grade 1, while others remain wholly unable to make sense of mathematical problems and operations through the last years of elementary school and into middle school. The teacher may face insurmountable barriers to developing the mathematical understanding of all the children in the classroom, and the difficulties are exacerbated in schools with "challenging" school populations that have a large admixture of low-income families, English language learners (ELL), and children

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with a variety of diagnosed or undiagnosed learning disabilities. The usual approach nowadays is to provide some sort of "differentiated" instruction that essentially provides different curriculum material to students who appear to have different capacities for thinking mathematically.

A more optimistic approach is taken in Davydov's mathematics curriculum (Davydov, 2008, chap. 5), which aims to develop all children to a high level of mathematical thinking. It is a "developmental" instruction that tries to develop all children's mental capacities. Children's different styles of learning are indeed taken into account, not by differentiating the content of instruction but by including all the children within various forms of collective discussion and peer interaction (whole-class discussion, small group work, work in pairs). All the children are exposed to the same, very high level of mathematical content, which at first view looks like high-school mathematics (use of algebraic notation, counting in bases other than 10). In particular, Davydov's curriculum has the goal of developing, in all students, a scientific, or theoretical, *concept of number,* from the very beginning of Grade 1.

The radical nature of this goal should not be underestimated. Davydov's approach provides a restructuring of both the *content* and the *form* of classroom learning. And the goal of forming a scientific concept of number in first graders — of developing their genuine conceptual thinking — is an exceedingly high and unusual one. Most curricula assume that if children can count when they enter first grade then they "know what number is," and move on to training the children in operations with number (starting with addition and subtraction of single-digit numbers). This basic approach to the start of mathematics instruction is almost universal, no matter how it is disguised by the use of manipulatives and visual aids or by different attempts at collective learning.

Traditional curricula forge ahead as rapidly as possible towards computational skill development, without really trying to check whether the children understand the reality that underlies mathematical operations and terminology. This raises the question of how truly to assess the level of development of the concept of number in children as early as Grade 1, no matter what curriculum they have been exposed to. In particular, does Davydov's curriculum indeed develop a scientific concept of number in first graders? And, if the concept of number has not been developed by the end of Grade 1, how can we further trace and support its development through Grade 2 and beyond?

Davydov's approach organizes the curriculum so that the children's thinking moves from the general to the particular, leading in principle to the immediate mastery of a broad class of problems or tasks. This is in distinction from all other existing approaches to mathematics instruction, which are based on training children in a large number of particular cases related to a rather narrow range of tasks, in the hope that the children will eventually build up a general understanding. For Davydov, a theoretical concept is itself a *general method of acting* — a method for solving an entire class of problems — and is related to a whole system of object-oriented actions. In a sense, the children learn from the very start to solve an infinite manifold of related mathematical tasks. If this is so, then the problem arises of how to assess the mastery of an infinite range of problems using a necessarily limited number of assessment exercises. In the present paper, I report on a first attempt at solving this assessment problem in a U. S. school, based on adapting and extending several assessments that were originally suggested by Davydov and his collaborators.

This work was the culmination of a project (Moxhay, 2003) over six school years (2002–2003 through 2007–2008) that tried to adapt and transform one version of Davydov's mathematics curriculum in the Portland Public Schools (Maine, U. S. A.). Here I will report only on some of the results obtained during the last two years of the project (2006–2007 and 2007–2008) and in just one of the participating schools, Presumpscot Elementary School. I will concentrate on these last two years of the project because it took the first four years of the project to train the teachers to a high level of understanding of Davydov's approach and to adapt the curriculum and the corresponding assessments.

It may be useful to give a brief description of the corresponding school population. Portland, Maine, is a culturally and economically diverse community with a significant number of recent immigrants. Presumpscot School has a 28% ELL population and 60% of its students qualify for free or reduced lunch. Its neighborhood is a mixture of subsidized and middle-class housing, and is described as a "multi-generational" neighborhood. The population of students is moderately challenging for instruction, but it is not quite the most difficult school population within the city of Portland.

The version of Davydov's mathematics curriculum that we tried to adapt was the most recent version (Gorbov, et al., 2001) intended for Russian four-year primary schools, i.e. schools where the children enter Grade 1 at age 6. An earlier version of the mathematics curriculum was originally developed for use with Russian children who began school at age 7, as had been traditional in the Soviet Union. The more recent Russian version shows significant accommodation for the younger age of the students, since in the past age 7 was considered the age at which learning replaced play as the leading activity driving children's mental development. We found that to use the curriculum in our school system we had to perform considerable cutting and reordering of the curriculum material, not only to make the lessons accessible to our students but also so that the mathematics covered was not too far off the schedule dictated by local and national standards and assessments.

I will not discuss Davydov's curriculum, or our adaptation of it, in detail here. But one of its essential aspects is that, in order to develop the concept of number, it delays the very introduction of number until several months into Grade 1. The lessons during the first trimester of Grade 1 concentrate on "pre-numerical" material: properties of objects such as color, shape, and size, and then quantities such as length, volume, area, mass, and amount of discrete objects (i.e. collections of things, but without yet using number to enumerate "how many"). The curriculum is structured in this way so that the concept of number emerges out of the concept of mathematical quantity as its precondition. In our schools, we made some cuts in the curriculum so that the introduction of number takes place no later in the school year than December 1.

Presumpscot School is a small school; during the years I will report on, each grade level consisted of only two classrooms, with about 20 children in each classroom. Although the project with Davydov's curriculum took place in other schools as well, it is useful to carefully analyze our results in developing the concept of number in this one school, because the level of teacher preparation was uniform, and of high quality, across the participating grade levels. (We used Davydov's curriculum throughout Grade 1 and Grade 2, and partially in Grade 3, for the topic of adding and subtracting multidigit numbers.) Presumpscot School was unique, within our school system, in that Davydov's curriculum was essentially adopted school-wide (in the relevant grades) and so avoided any conflict between teachers using Davydov's approach and those using more traditional curricula.

## II. SOME ASPECTS OF CHILDREN'S CONCEPTUAL THINKING

### Conceptual thinking: object-oriented, generalized, and reflective

To understand how and why we chose particular assessments for the concept of number in first- and second-graders, it is necessary to discuss Davydov's very idea of what a concept is, and what are the conditions for developing the concept of number in primary school children. From this discussion, it should become clear that the very notion of a concept has been much more fully worked out in Russian psychology and philosophy than in the Western pedagogical literature. Davydov's "concept of number" goes far beyond "number sense" or "numeracy" as an assortment of desired abilities and skills with numbers.

In Davydov's approach, stating that children have a concept, or that they manifest conceptual thinking, has to be based on the presence of three properties. First, their thinking must be *object-oriented,* that is, it must be primarily grounded in transformational actions with real objects rather than in actions with signs. Davydov writes (Davydov, 2008, chap. 4):

> …every concept conceals a special cognitive, object-oriented action… that must be discovered in order to reveal the psychological mechanisms for the emergence and functioning of the given concept.

Second, it must be *generalized* — it must relate to a whole system of related tasks (problems) rather than being based on training the children to solve a narrow range of particular tasks. Third, it must be based on the thinking action of *reflection* — the capacity to consider the basis of one's own actions (related to the idea of metacognition as the ability to think about one's own thinking). Let us consider each of these three properties in turn.

## Object-oriented thinking

If conceptual thinking is, in a primary sense, object-oriented, i.e. if it emerges from object-oriented actions or activity, then to study the formation of the concept of number we need to determine what is the *object* of school mathematics. (In this paper, by "mathematics" we will mean arithmetic, i.e. the study of number. We put aside the question of developing the concepts of geometry, for example.) Another way of posing the question of the *object* of mathematics is to ask what human *task* is solved by mathematical methods of acting that arose in the course of human history.

To fix some terminology, let is first consider a different task or puzzle, the "nine dots problem" (Figure 1).

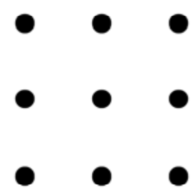

Figure 1. The nine dots problem

In this task, the goal is to place the point of your pencil down on the paper and then, without lifting your pencil from the paper, draw four straight lines in such a way that the resulting path goes through all nine dots. A typical attempt at solving this task might look like this:

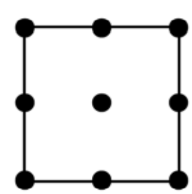

Figure 2. The nine dots problem: a failed solution

The person trying to solve this task usually makes several analogous attempts and then gives up. It just does not seem possible to solve the task using only four lines.

There is, however, a solution, which looks like this:

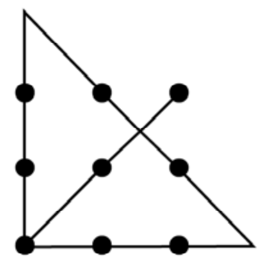

Figure 3. The nine dots problem: a correct solution

The point here is not that this is a “trick problem” or “brain teaser,” but that any task that must be solved has not only a *goal*, but also certain *conditions* under which the goal has to be attained. Initially, the conditions were ambiguous. We knew that we had to draw just four lines, and that they had to be straight lines that were connected, but it was ambiguous whether or not those lines had to stay within the square formed by the nine dots. The creative aspect or moment of solving this task was the person’s *transforming the conditions* of the task in such a way that a means and method for its solution emerged. In a sense, the person had to “break the rule” (even though going outside the square was a self-imposed, unwritten rule) in order to create a method of solving the task. In this case, the conditions were transformed from their initial ambiguity into a clear understanding that we can draw the lines outside of the square formed by the dots. Such a transformation of the conditions of the task is an aspect of any truly creative problem solving and plays an important role in Davydov’s writings about children’s learning.

What is the *object* being acted upon in the nine dots problem? We might say that it is not only the nine dots on the pages, and not just the pencil and the straight lines we draw, but also that the very conditions of the task are included in the object we act on.

So, what is the object of mathematics (arithmetic)? What is the goal of the task that was solved by the invention of number during the course of human history? In answering this question so as to construct a school mathematics curriculum, Davydov and his collaborators are not interested in the details of how the use of number arose during, say, the Bronze Age, but in a concise logical distillation of that historical process. The result of their work is that mathematics can be regarded as a response to the task of taking a given quantity (length, volume, mass, area, amount of discrete objects) and reproducing it at a different time or place.

The corresponding task can be posed in the classroom in a variety of ways. In one version, the children are asked to reproduce a line segment drawn on the blackboard at one end of the room on a blackboard at the other end of the room. In another version, they might be asked to determine ahead of time whether there is room for some heavy piece of furniture, say a bookcase, to be moved into some limited space (say, between a desk and a window) in another room. Let us illustrate this task, whose solution can lead to the children’s discovery of number, using the following materials (Figure 4).

Figure 4. The learning task that leads to the discovery of number

On one table, there is a strip of paper tape. The task is to go to the other table (in a different room) and cut off, from the supply of paper tape, a piece that is exactly the same as the original one, that is, exactly the same length. But there is a strict rule — we are not allowed to carry the original paper strip over to the other table. How is the child, or the classroom of children, going to solve this task? A child will often just walk over to the second table and cut off a piece of paper of a random size, just hoping that it is the same length as the original one. Upon bringing it back and comparing with the original, the length turns out to be wrong. To such a child, conditions of the task seem to make it impossible to guarantee its correct solution (except by luck). How can it be possible to get a piece of exactly the correct length, every time and without fail? The other table where we have to cut the new paper strip is too far away, so we can never be sure of getting it to be just the right length!

Here, again, a correct solution can only be obtained by *transforming the conditions of the task* in some way. In this case the solution is obvious to an adult who possesses the concept of number, but is not at all obvious to students in Grade 1 or Grade 2. The children need to find the solution through group discussion and by trial and error. The solution may take many forms, but always involves "breaking the unwritten rules" and bringing in something new, in this case, some third object that can be used as an intermediary.

One solution (Figure 5) might involve taking some third object, such as a string, and cutting it to be just precisely the length of the paper strip, and then carrying this intermediary object (the string) to the other table, where it can be used to lay off a new paper strip of the required length. In this case, the intermediary is equal in length to the object to be reproduced.

Figure 5. Solution with an intermediary (variant 1)

In a different solution (Figure 6), the children might take a given third object, say a piece of wood, and mark it so as to show the length of the paper strip. In the case illustrated, the available intermediary is longer than the object to be reproduced, but the action of marking it makes it work for the task at hand. So this solution is equivalent to the first one, with the children performing just a different set of operations.

Figure 6. Solution with an intermediary (variant 2)

What if the only available intermediary object is *smaller* than the paper strip? For example, it might be a wooden block (Figure 7). This is the most interesting case of all, for then the children need to discover that they can use the block as a *unit* — as an intermediary than can be placed repeatedly (each time marking the paper with a pencil), and then *counting up* how many times the unit has been laid down. The unit is then carried to the other table (together with the number) where it is laid down on the paper tape the number of times that is necessary to reproduce, by cutting, a paper strip of the required length.

Figure 7. Solution using a unit

According to Davydov, by means of such a discovery, the children recreate, in brief, the invention of number as a human tool that enables *any* quantity to be reproduced at a different place or time. (This discussion is very condensed — in the actual lessons in the Davydov mathematics curriculum, this discovery is drawn out in a step-by-step manner over the course of several weeks.) In a sense, this discovery is the key that can lead to all the children in the class developing the concept of number.

Although this task with the paper strip is a particular task, solved by a particular discovery on the part of the children, it leads "genetically" to the solution of all analogous tasks. If the children, working as a collective, grasp the meaning of the discovery they have made, then they should (again, collectively, at least at first) be able to attack all analogous problems. The key point here is a special relation between the universal and the particular and individual. This relation is completely nullified if the teacher thinks a concept is something that is verbally formulated, or if the teacher merely "hands" the children the solution. The children absolutely must participate in transforming the conditions of this task to discover the means for its solution.

To summarize, the children's solution of the task described above is a manifestation of their *object-oriented* thinking. What is important for concept development is not "hands-on" work *per se*, but the creativity that results from the children's transforming the *conditions* as a result of their collective discussion and trial-and-error work. The conditions are the most important thing for the children to "get their hands on." If this circumstance is missing, then there will be no scientific (i.e. theoretical) concept development. Indeed, this is a widely misunderstood aspect of the use in instruction of so-called manipulatives, which like any visual aid are only capable of forming superficial, empirical concepts.

Moreover, in designing an assessment of the concept of number, the focus must absolutely be on a child's ability to perform just this sort of object-oriented action, where the use of number is required, but where no concrete number is referred to in the statement of the problem. *Number* as a means for solving the problem has to be introduced by the child, it should not always be provided by the designer of the assessment.

According to Davydov, solving this sort of specially designed task provides, in principle, a *general method* for solving all problems within a very broad class of problems — in this case, all problems involving the use of number! But in what sense have the children achieved a general solution? What role is played by generalization, and what kind of generalization are we talking about?

## Generalized thinking

Thus, it is natural to ask whether the children, once they have solved the task with the paper tape described above, can then solve any analogous task. Most likely, they will not. As an example consider a task involving volumes of water (Figure 8). This task assumes that (as in the pre-numerical stage of Davydov's curriculum) the children have already mastered the action of directly comparing the volumes of two containers, for example, by filling one with water and pouring the water form one container into the other one.

Figure 8. Analogous task with volumes of colored water

In this task, at one table we have a given volume of colored water, and the goal is to go to the table in the other room and pour off an equal volume (from the pitcher or some other supply) into the other, empty container (which happens to be of a different shape). Once again, the children can solve this task by introducing a third object (a small cup, as in Figure 9) that can be used as a unit volume.

Figure 9. Introduction of a unit volume

We can measure the given volume by repeatedly pouring into the unit container and counting up the number of units, and then carry the unit to the other table and use it to pour off an equal number of units, thus reproducing the required volume in the container at the other table.

Clearly, we could create an infinite set of tasks equivalent to the two tasks just described. Many children, or classrooms of children, will approach the task with the volume as something completely new, even after having mastered the task with the paper tape. Others may immediately generalize from the first task to the second, and such a generalization will be more and more likely as the age of the children increases. How is it possible to enable children in Grade 1 or Grade 2 to form such a generalization? If they can form this generalization, and solve all tasks of this infinite set of tasks, then they may be said to have formed the *concept of number* as a general capacity from which there is no turning back.

In Davydov's approach, this process of generalization is aided by the use of a special *model* to represent the measurement process, one form of which is shown in Figure 10.

$$A \xleftarrow{6} E$$

Figure 10. The model of the measurement process

In this model, the arrow represents the process of measuring with the unit. One of the capital letters (which can be chosen arbitrarily, although here it is *E*) represents the unit (of length, of volume, of area, etc.) while the other capital letter (here chosen to be *A*) represents the quantity that is to be first measured and then reproduced. The unit letter is always placed at the tail of the arrow (which can point in any direction), while the letter for the measured quantity is placed at the head of the arrow. In our two examples the unit *E* is in one case the length of an edge of the block or the volume of the unit cup, and *A* is the length of the paper strip or the given volume of water. Finally, the numeral written above the arrow represents the number of times that the unit was used in order to measure the given quantity.

The significance of the model is that it is capable of replacing any task within our system of related tasks in measuring or constructing a quantity. It can replace a measurement task with length, with volumes, with areas, and so on, and abstracts from all the materials used and from all the different operations that must be applied.

The children come fairly quickly to master the model by applying it to a small number of measurement tasks, and this supposedly enables them to form a generalization — to have a general method of solving all such "concept of number" tasks (i.e. measurement tasks). In designing an assessment of the concept of number, a role may be played by assessing the children's mastery of a model as an auxiliary probe of the degree to which they have generalized the object-oriented tasks and can see them as particular cases of a general class of tasks.

## Reflective thinking

One fundamental aspect of the children's thinking that we have not discussed, an on which Davydov places great emphasis, is metacognition or *reflection*, defined as the capacity to understand the basis for one's own actions. Strictly speaking, there can be no concept of number if the children are not fully aware of their own discovery of it. Not only must they have transformed the conditions of the learning task described above, but they should be aware (conscious) of what they did, and why, and eventually they should be able to clearly articulate the use of the unit, the role played by the number, and so on.

Such an ability to reflect on one's own actions is generally beyond the capability of first- and second-graders, and yet the children must be able to do this in order to truly understand number or any other system of tools or signs they are ask to work with in school. Reflection can be thought of as both the starting point and the end result of the process of developing a concept, in this case the concept of number. This is possible, in Davydov's curriculum, due to the collective nature of the inquiry process. The children "pool their mental resources," as it were, in the process of solving the learning task. In the course of whole-class discussion, the teacher guides to children to *ask each other questions* about the process of discovering number — "How did you do this? Why did you do this? Does your method work? Is it the most suitable method for solving the task?" Eventually, this dialogue among students becomes an internal dialogue, so that the individual child is eventually able to reflect on the basis of his or her actions (i.e. thinking).

An assessment of the concept of number should therefore include some way of evaluating the children's development of reflective thinking — possibly that of individual children but if that is impracticable then that of the class as a whole.

## Some errors regarding the concept of number

The most common error in evaluating the child's concept of number is to regard the child's being able to count as demonstrating that the child understands number. From the object-oriented tasks described above, and from the model (Figure 10), it is clear that counting is only one of the operations that make up the action of measurement that underlies the concept of number. To focus only on counting omits the most important thing — to be able to single out and use a unit, that is, to be able to see number as arising from a ratio relation between two quantities.

Another common error is to give the child only tasks involving some countable set of discrete objects (blocks, pictures of fish, plastic counting chips, etc.). Such tasks prompt the child to assume that the unit is has to be a single such object, and nothing else, and in effect "hides" the very presence of a unit. Moreover, tasks that use only collections of discrete objects do not probe whether the child has formed a generalization — whether he or she can see a number or ratio relation within all such analogous tasks. It is essential to use several tasks using lengths, volumes, and, if practicable, areas and masses, as well as collections, in assessing the child's concept of number.

Finally, it is an error to regard the object-oriented tasks described above as relating not to "number" but to the curriculum topic of "measurement." It is if course true that, according to Davydov, the action of measuring using a unit is the action that underlies the concept of number. But what is essential in these tasks is that they recreate the very emergence of number, in human history, as a tool needed to reproduce a given quantity, in response to a particular human need. Moreover, the "nonstandard units" used in these tasks are not a generalization from standard units (inches, liters) that the children have used previously when studying measurement. They emerge from the conditions of the task and from the children's transformation of those conditions. It is in the emergence of something new out of some given preconditions, and in response to a need, that a concept (in this case, the concept of number) can be observed most clearly. Davydov's identification of a way to simulate the historical emergence of number in the classroom is of the greatest significance for mathematics pedagogy, and should not be trivialized by identifying it too closely with "measurement" as a topic that is distinct from "number."

## III. THE ASSESSMENTS

Our selection of assessments of the concept of number was driven by two main considerations. The first has to do with how to test a *general* object-oriented method of acting. The paradox is that we would like to test whether the child can solve any task within an infinite manifold of related tasks. Perhaps the child can solve tasks *a, b,* and *c,* but then suppose we present him or her with an equivalent task *d*? How in the world can we ever be sure the child has really mastered a general method? This must be considered an open question. The best response we could come up with was to test the child using a handful of object-related tasks, and then, in addition, to pay particular attention to assessing the child's mastery of the *model* (Figure 10) as something that can replace all tasks within the potential infinite set of tasks.

The second consideration in selecting the assessments had to do with the age of the children (as young as Grade 1, i.e. 6–7 years old). The object-oriented tasks must be comprehensible to first- and second-graders, but several years experience enabled us to choose appropriate tasks. It was a different matter with regard to assessments that used the model. The assessments had to be especially carefully designed in the case of first graders, since lengthy assessments, especially of the pencil and paper type, may not be appropriate. Moreover, we found that our children often had limited graphical ability and found it difficult to perform some of the Davydov curriculum tasks that involved drawing areas, for example, to perform a measurement. We ended up choosing a very simple set of written tasks and supplemented this with measurement tasks performed using computer software that we created.

Thus, we ended up with three different assessments, and decided to look at the correlation among the results to form a judgment as to the level of development of the children's concept of number.

(1) Object-oriented tasks

We selected four object-oriented tasks related to some of those that were suggested by Davydov himself (Davydov, 1990, pp. 144-145). These were administered by an adult working one-on-one with the child. We transformed the tasks somewhat by choosing materials and situations that our children could understand at any grade level. These four tasks are described in detail in Appendix 1. Briefly, Task 1, with paper strips and with volumes, is related to the action of reproducing a quantity, as described in detail in Section II of this paper. Task 2, with large and small cups, tests the child's ability *not* to be guided by what is visually obvious (four cups means the number 4) but to analyze the relation between the unit and the quantity measured. Task 3 tests the child's flexibility in assigning *any* number to some countable collection, depending on what unit is indirectly suggested by the adult. Task 4 tests the child's ability to analyze what numbers will result when a given quantity is measured using two different units.

(2) Written tasks

The first of the two written tasks is shown in Figure 11. This tests the child's ability to measure a collection of objects using a given unit and to model the result using an arrow diagram. Here it is extremely important that the child places the letter for the unit at the *tail* of the arrow, and that the correct number of units is written above the arrow.

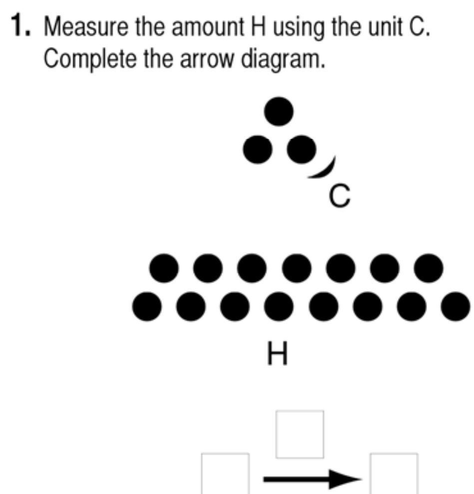


Figure 11. First of two written tasks. The little arc is a conventional sign to indicate the unit of measurement

The second of the two written tasks is shown in Figure 12. It is described in a book by two of Davydov's mathematics curriculum writers (Gorbov & Tabachnikova, 1995, pp. 32-34). This task tests the child's ability to reflect on the very basis for being able to measure (in this case an area) using a unit. In the first case, the answer is simply $A = 4\ E$. In the second case the child should reject the problem as a trap, since a whole number of units $E$ does not fit into the area $B$. (Gorbov and Tabachnikova describe several possible adequate responses of the child to this task.) In the third task the child should understand that the unit $E$ may be rotated, so that three units $E$ fit into the area $C$; this problem is not a trap.

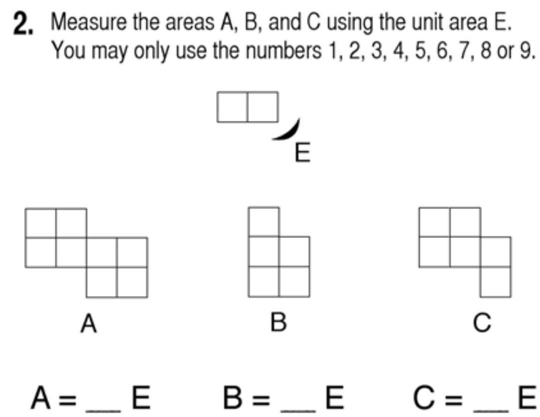


Figure 12. Second of two written tasks

(3) Computer tasks

To complement Davydov's mathematics curriculum, we created a sequence of 50 computer programs (java applications) to enable children to practice various object-oriented and modeling actions as well as to practice number skills. Programs 1 through 8 pertained to the "pre-numerical" stage of instruction, while Programs 9 and 10 related to measurement, i.e. to the concept of number. Program 9 is illustrated in Figures 13 and 14. Program 10 (not illustrated) essentially reproduced Written Task 2 (Figure 12).

The programs were gradually introduced during the course of the curriculum. Students were able to use the programs on a weekly basis during a "computer lab" time slot. Students were required to do the programs in sequence, e.g. they were not allowed to go on to Program 9 until they had successfully completed Program 8. A student was considered to be successful at a given program if he or she was able to perform 20 problems with 80% of the corresponding actions done correctly.

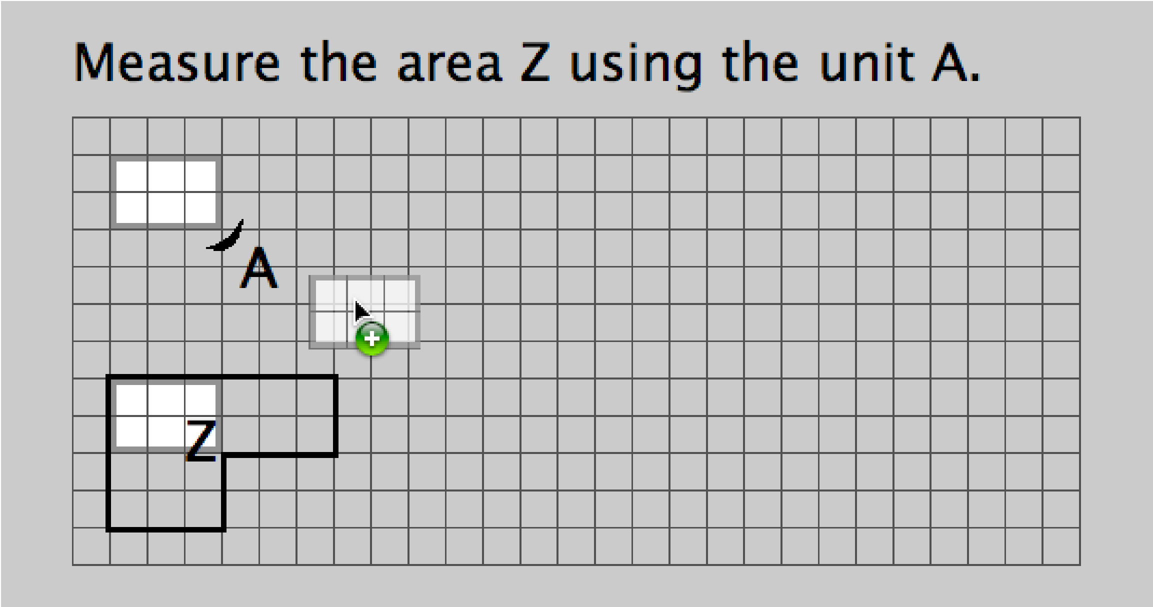


Figure 13. Computer program 9: measuring an area using a unit

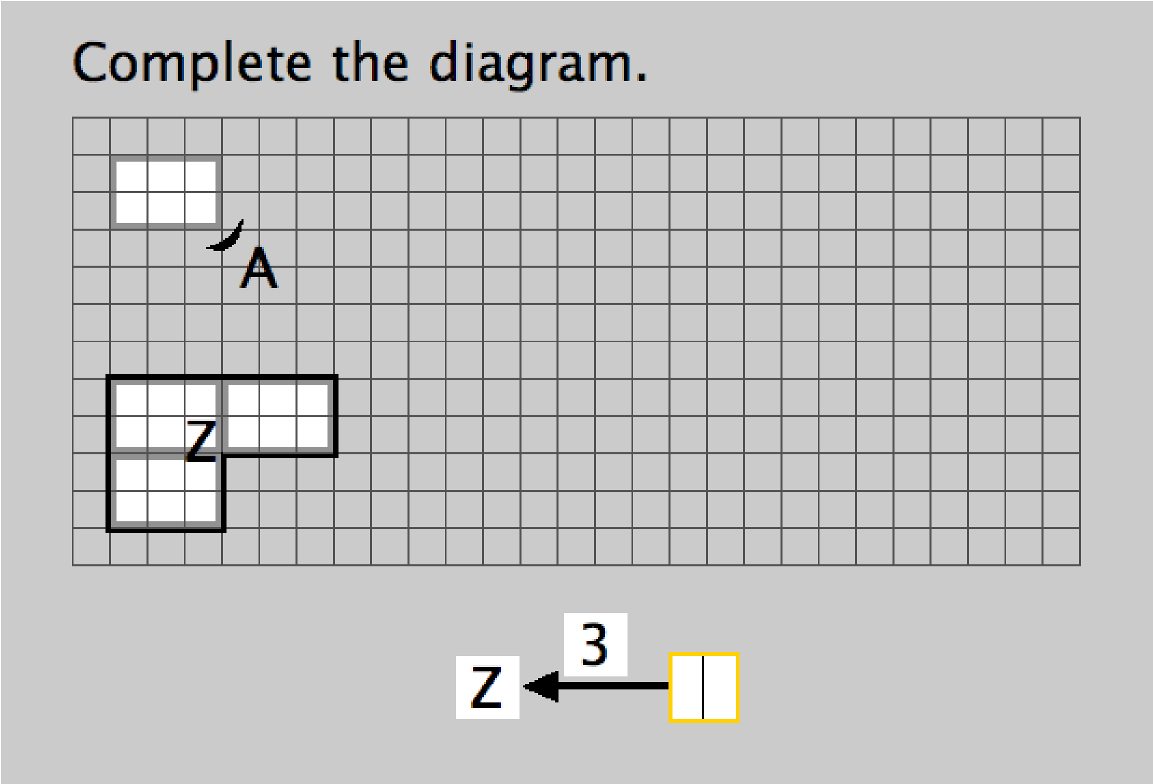


Figure 14. Computer program 9: completing the model

(4) Zak's test of levels of theoretical thinking

Besides these three proposed assessments of the concept of number, we decided that it was important to assess the emergence of the children's reflective thinking. After all, as discussed above, reflection is a prerequisite as well as a result of using Davydov's curriculum, and is also a component of conceptual thinking.

To assess the children's thinking, independent of the mathematics curriculum material, we used a test (Atakhanov, 2000, pp. 126-128) originally designed by A. Z. Zak (Zak, 1984). This test involves the children's ability to perform a series of verbal deductions, such as:

> 1. Mike is noisier than Jim. Liza is quieter than Jim.
> Who is the quietest of all?
>
> 5. Hvfn is ardk than Dmln. Vkpt is ardk than Hvfn.
> Who is ardk of all?
>
> 8. The horse is shorter than the fly.
> The horse is taller than the elephant.
> Who is the tallest of all?

To avoid any misconception, here the problem is not at all whether the children can attempt such deductions — it is not to test their deductive thinking, which we have found to be well within the capabilities of children at this age. Rather, it is to test whether the children can see that all the three cases given above (and other cases as well, including more complicated ones, for a total of 12) are essentially the same. We found this version of Zak's test to be entirely suitable for children at the end of Grade 2, when most of the children are starting to read and write fairly well. (The problems were read aloud once or twice by the teacher while the children read them silently.) We did not have any difficulties in using Zak's test with the ELL students in our particular population at Presumpscot School.

The version of Zak's test we used (one of several variants) is presented in Appendix 2. If the children were able to do all of problems 1 through 8 correctly, they were assumed to possess the thinking action of contentful analysis. If they were also able to do the multipart problems 9, 10, and 11, they were assumed to possess the thinking action of planning. If they were, in addition, able to attack problem 12:

> 12. Tom is stronger than Zack. Brad is weaker than Tom.
> Who is the weakest of all?

and conclude that this problem has no unique answer, they were assumed to possess the thinking action of reflection. Again, we anticipated that very few second graders would possess reflective thinking, but that the collective results for the entire class might indicate an early development of theoretical thinking for the class as a whole.

## IV. RESULTS

The assessments were applied to 36 Grade 1 students at the end of the 2006–2007 school year and 38 Grade 2 students at the end of 2007–2008. In each year, there were two classrooms of about 20 students each; some students have been omitted from the data because they moved in or out of the district early or late in one of the school years. There was about a 12% turnover rate between the two school years, but children who participated for all of Grade 2 but none of Grade 1 experienced no difficulty in adapting to the curriculum, so we included them in the Grade 2 data.

All of the data are presented in terms of the percentage of tasks solved correctly by the children in the corresponding grade level. As a consistent convention for referring to the results in the graphs, the correct scores are labeled "Theoretical" and the incorrect ones "Empirical." This makes some sense, since all of the children, even in Grade 1, were able to count up a collection of objects and name the number correctly, and so possessed at least an empirical concept of number.

Figure 15 gives our results for the end of Grade 1. These assessments were done in May 2007 at Presumpscot School in Portland, Maine. The data for the two classrooms were combined because in this case the students had the same teacher for math and because there was no appreciable difference between the classrooms in terms of the children's acceptance of the curriculum material and style of instruction.

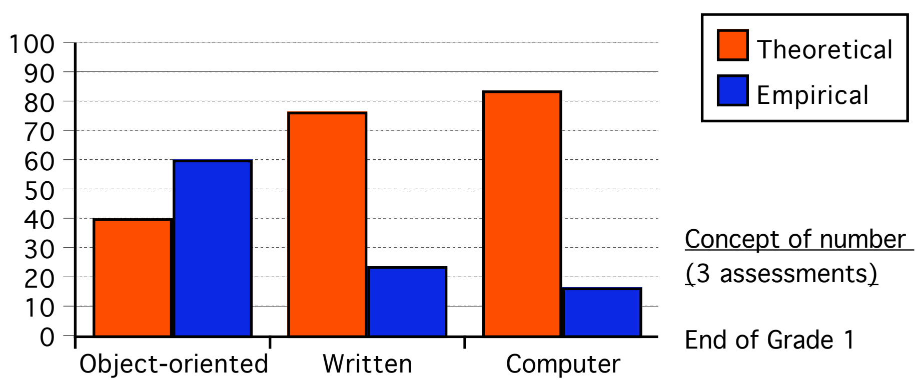


Figure 15. Percentages of tasks performed correctly ("Theoretical") and incorrectly ("Empirical") for 36 students at the end of Grade 1

One might say that here the main result is that, after one year of instruction using Davydov's curriculum, at best 40% of the students emerged with a scientific or theoretical concept of number. On the other hand, the graphs for the written and computer assessments show that the students did somewhat better in assimilating the model of the measurement process. On the whole this makes sense, since learning the model (a sign) should precede a generalized assimilation of the object-oriented actions — the model should be a tool for forming the corresponding generalization.

Figure 16 shows the results of assessing 38 students at the end of Grade 2, during May 2008. In this case, I present the data for the two classrooms, here referred to as Classroom A and Classroom B, separately. The reason for this is that the classrooms were quite different in terms of student behavior and so different results were obtained.

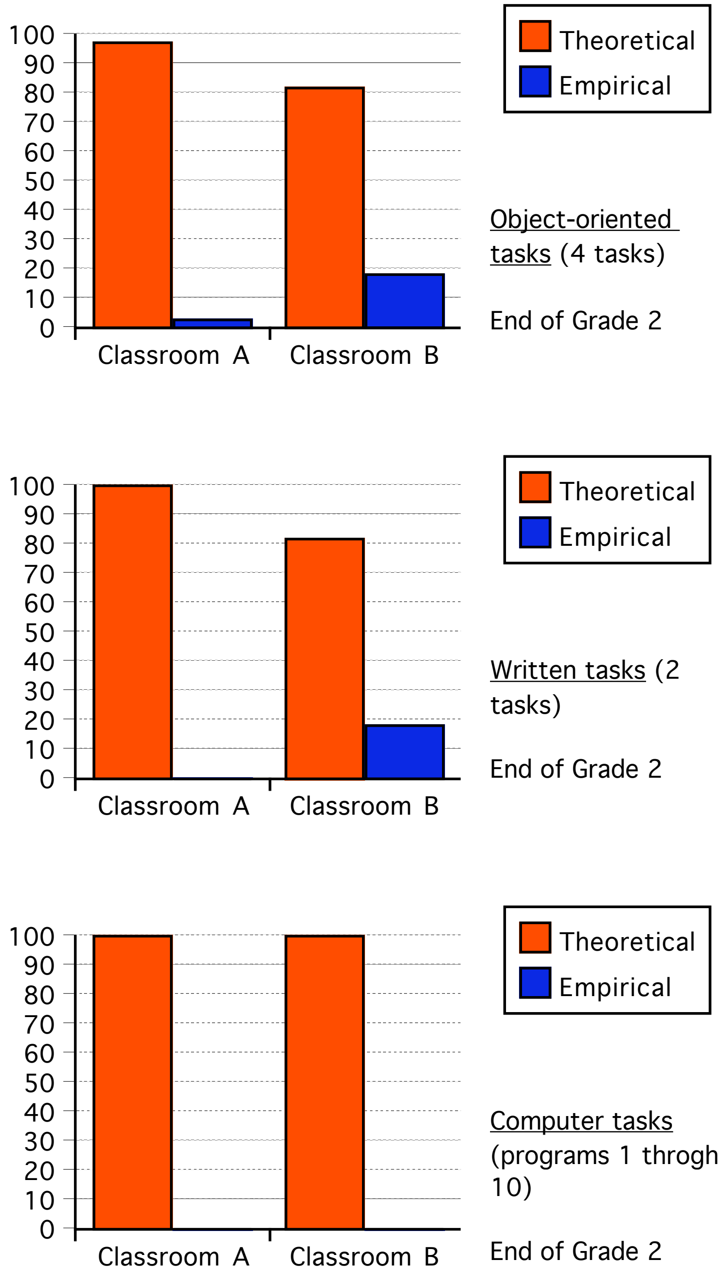


Figure 16. Percentages of concept-of-number tasks performed correctly ("Theoretical") and incorrectly ("Empirical") for 38 students at the end of Grade 2

Both classrooms used Davydov's curriculum, with most of the children in both classrooms having had the curriculum in Grade 1. In Classroom B, however, the behavior of a number of children made it very difficult for the teacher to implement the type of inquiry demanded by the curriculum, and so we can regard this classroom as a kind of control group in which the *content* but not the *form* of instruction was implemented. In effect, the students in Classroom B rejected the Davydov style of inquiry where the teacher acting mainly as a facilitator of the students' interaction. This was despite the fact that the teachers in both classrooms had already worked well with the curriculum for four or five years and, in my evaluation, had mastered the approach equally well. In contrast, both the content and the form of instruction were implemented very successfully in Classroom A.

We can draw the provisional conclusion (given the provisional nature of our set of assessments), that a scientific concept of number was successfully formed in close to 100% of the students in Classroom A, but that this failed to occur in Classroom B.

At the end of Grade 2, we tested not only the second-graders' development of the concept of number, but also their subject-independent theoretical (reflective) thinking, using Zak's test. For simplicity, the graphs do not distinguish among Zak's three levels of theoretical thinking. The results are shown in Figure 17.

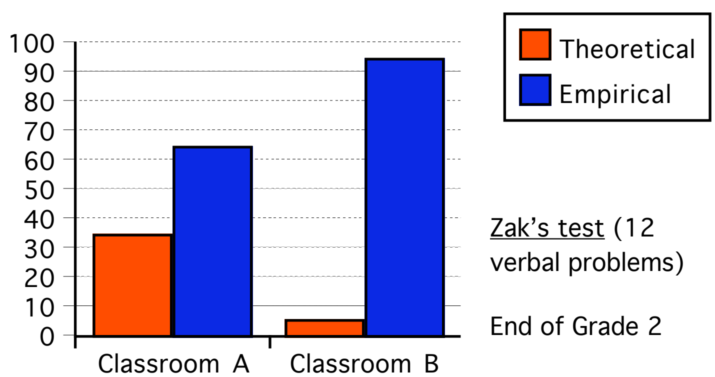


Figure 17. Percentages of students scoring at the "Theoretical" and "Empirical" levels, for the 38 students at the end of Grade 2, on Zak's test of theoretical thinking. In Classroom B, only a single student scored at the theoretical (planning) level

Zak's test shows a clear distinction between the levels of thinking in the two classrooms. This is perhaps an indicator of the developmental effect of Davydov's curriculum in Classroom A, where the form of instruction really "took," but such a conclusion would require a more detailed study of the development of the children's thinking from the start of Grade 1. The lack of development of the children's subject-independent theoretical thinking (reflection) in Classroom B throws doubt, in my opinion, on whether we can even say that those children have a scientific concept of number at all, despite their adequate performance on thee assessments that used mathematical material.

## V. CONCLUSIONS

First of all, I would like to emphasize that we have made only a first attempt at assembling a viable assessment of the concept of number. A reasonable correlation among our three assessments shows that we are not too far off. Moreover, the assessments are based on the profound theoretical analysis and vast empirical experience of V. V. Davydov and his colleagues. But further work clearly needs to be done, and I would welcome criticism of these assessments and suggestions for other, more effective ones, and for our assessments to be tried with other populations of students.

If these assessments are indeed along the right lines, then our results for the end of Grade 1 are rather dramatic, despite the small sample size. In the school population we dealt with, we may say that it was not possible to fully develop a scientific concept of number in first graders — even in first graders using Davydov's outstanding mathematics curriculum and with a well-prepared and experienced teacher. If this is at all indicative of the general population of children age 6–7, it implies that Grade 1 students are rather unlikely to "really understand number," and should not be forced into rote memorization of addition and subtraction facts and other superficial approaches to mathematics learning. It implies that virtually all existing Grade 1 mathematics curricula push children far beyond what they are capable of understanding deeply.

It was possible, however, to develop a scientific concept of number by the end of Grade 2. But the success rate was close to 100% only in the classroom where the children accepted Davydov's form of instruction. In the classroom where apparently behavioral issues derailed this nontraditional type of instruction, any true development of the concept of number is cast into doubt by our assessment results.

If some children do not develop a genuine concept of number by the end of Grade 2, then when does it develop? If used in Grade 3 and up, our assessments may help in answering this question. Of course, number does not end with whole number. Davydov-like "interventions" may help older school children develop the concept of rational number (fractions and decimals), but merely exposing the students to classroom activities based on measurement are not enough. Such interventions need to be evaluated using assessments of whether the students really have *general* as well as *object-oriented* methods of acting with fractions, and whether their *reflective* thinking is sufficiently developed. If any of these three aspects of conceptual thinking is absent, there is great risk of the student's knowledge being reduced to a superficial, empirical level.

# APPENDIX 1: THE OBJECT-ORIENTED ASSESSMENTS

Task 1. *Version 1*. At one table: a strip of paper tape, pencil, and wooden cube. The wooden cube fits some integer number of times (6, 7, or 8) along the paper strip. (Don't tell the child this!) At another table: a spool of paper tape, scissors, pen. Tell the child that the problem is to go to the other table and cut off a piece of paper tape "just like this one," but the rule is that the only thing we're allowed to carry from this table to the other one is the wooden cube. The task is considered to be performed successfully if the child uses the wooden cube as a unit of length, measures the paper strip (marking each unit using the pencil), carries the wooden cube to the other table, and uses the wooden cube to lay off the required number of units.
*Version 2*. A similar task, but where the quantity to be reproduced at the other side of the room is a volume of water and the unit provided is a small cup. The child must pass both versions to be considered to have passed Task 1.

Task 2. You have small cups, large cups, and a pitcher of colored water. Ask the child to check to see how many times we have to pour from the small cup into the large one in order to fill it. (Two times, it turns out.) Then show the child a row of cups: 2 large and 2 small. Ask the child how many times we would need to pour from the small cup in order to fill the whole row of cups. The child should give the correct number (6, in this case) without having actually to pour. (You can vary the number of cups for different children: 2 large and 3 small, or 3 large and 2 small.)

Task 3. Place a collection of 12 shapes (red circles or green triangles, for example) on the table in front of the child. Then hold up set of 3 shapes. Say, "How many of *these* (pointing to the set of 3) do you have? (Pointing to the set of 12). Then show the set of 2, and then the set of 4. In each case the child should say the correct number (4, 6, 3). It is acceptable if the child takes until the second set of triangles to realize what is being asked — but do not vary the wording of the problem. The task should be considered *not* to have been solved if the child just says each time that there are 12 shapes.

Task 4. *Version 1*. Show a coiled up length of wire. (Do not uncoil it.) Say that someone stretched out the wire and measured it using first one of the cardboard strips and then the other one. (Show the red and orange strips.) Say that in one case the person got the number 7 and in the other case the number 5. Ask the child to figure out which number resulted when which cardboard strip was used to measure the wire. (The child should say that 7 went with the shorter, orange strip and 5 went with the longer, red strip.)
*Version 2*. Show the transparent cylinder filled (up to the mark) with colored water. Say that someone measured the water first using one of the small containers and then the other one. (Show the purple and green cups.) Say that in one case the person got the number 4 and in the other case the number 10. Ask the child to figure out which number resulted when which small container was used to measure the water. (The child should say that 4 went with the larger, purple container and 10 went with the smaller, green container.) The child must pass both versions to be considered to have passed Task 4.

APPENDIX 2: ZAK'S TEST OF 3 LEVELS OF THEORETICAL THINKING

1. Mike is noisier than Jim. Liza is quieter than Jim.
Who is the quietest of all? ___________
2. Val runs a little faster than Pat. Ella runs much more slowly than Pat.
Who runs the fastest of all? ___________
3. Andy is gbdt than Sam. Bob is gbdt than Andy.
Who is gbdt of all? ___________
4. Sptv is sadder than Ldvk. Sptv is happier than Mnpr.
Who is the saddest of all? ___________
5. Hvfn is ardk than Dmln. Vkpt is ardk than Hvfn.
Who is ardk of all? ___________
6.Dan is older than Elin. Elin is older than Meg.
Who is the youngest of all? ___________
7. Kalli is lighter than Drew. Jane is lighter than Kalli.
Who is the heaviest of all? ___________
8. The horse is shorter than the fly.
The horse is taller than the elephant.
Who is the tallest of all? ___________

* * * * * *

9. Sam is younger than Rick. Bob is lighter than Sam. Rick is younger than Bob. Sam is quieter than Bob. Rick is heavier than Sam. Rick is noisier than Bob.
Who is the heaviest? ___________
Who is the oldest of all? ___________
Who is the quietest? ___________
10. Anna is shorter than Mary and stronger than Nina. Anna is taller than Nina, but Mary is stronger than Anna.
Who is the shortest of all? ___________
Who is the strongest? ___________
11. Sally is happier and older than Lucy. Lucy is heavier than Sally, but Rita is older than Sally. Rita is lighter than Sally and sadder than Lucy.
Who is the youngest? ___________
Who is the lightest? ___________
Who is the happiest? ___________

* * * * * *

12.Tom is stronger than Zack. Brad is weaker than Tom.
Who is the weakest of all? ___________

Questions 1–8 correct implies analysis level of theoretical thinking (lowest)
Questions 1–11 correct implies planning level of theoretical thinking (middle)
Questions 1–12 correct implies reflection level of theoretical thinking (highest)

Any of Questions 1-8 incorrect implies only empirical level of thinking

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