THE QUARTERLY NEWSLETTER OF THE

LABORATORY OF COMPARATIVE HUMAN COGNITION

Center for Human Information Processing
University of California, San Diego

Volume 11, Numbers 1 & 2 January/April, 1989

TABLE OF CONTENTS

Introduction

The contributions to this issue of the *Newsletter* seek to formulate an adequate approach to understanding of one of the most complex and richest domains of human activity: mathematics. The main articles were presented at a symposium organized by our fellow editor, Terezinha Nunes Carraher, at the 1987 meetings of the International Society for the Study of Behavioral Development in Tokyo. A commentary on the symposium contributions is provided by Peter Bryant and the issue concludes with a book review by Derek Edwards.

Readers of the *Newsletter* will recognize many familiar themes in these articles:

(a) approaches to understanding the development of mathematics that take their inspiration and often their empirical starting point from the structure of institutionalized activity—whether in a LOGO lesson, an abacus class, a standard math lesson, or the selling of candy;

(b) a focus on mediational means, whether computer, abacus, paper and pencil, or money;

(c) a concern with social-organizational factors that constrain activity in context, differentiating (for example) everyday and school-based mathematics.

In the discussion that Peter Bryant was kind enough to write for this issue, he raises a problem that has long been of concern: how are we to account for children's successes and failures in bringing mathematical knowledge they display in one context to bear on problems encountered in another? Bryant suggests that a key to transfer is *analogy*, a process by which children come to see that two problems in different contexts in fact share the same underlying cognitive structure.

Bryant is correct when he comments that an appeal to analogy represents a departure from what he takes to be existing "causal models" of development: Piaget's and Vygotsky's. Socio-historical theorists, at least, have long been suspicious of the appeal to analogy as the basis of transfer since such an appeal names the problem instead of solving it; the issue of how children are able to see two problems as "isomorphs" of each other is just the question of how transfer is possible. Interested readers are invited to refer to an extended discussion of this issue in LCHC (1983). There we point to a variety of ways in which socially organized connections between contexts play a role in highlighting cultural assumptions about orders of relevance between problems as "extra cortical" constraints on the discovery and use of analogy.

A complementary source of constraints on the discovery of relevant analogies is suggested by Derek Edwards in his review of Valerie Walkerdine's *The Mastery of Reason*. Edwards suggests that one fruitful way to understand how analogies are formulated and deployed is to focus on modes of discourse which contribute to the constitution of activity in context. Indeed, many of the concrete results reported in articles given at the Tokyo symposium rely on analyses, both implicit and explicit, of children's discursive accomplishments as they deploy mathematical procedures in a variety of contexts. This approach to discourse appears fully congruent with recent efforts within the socio-historical tradition to elaborate the semiotic dimension of activity-centered approaches to mind (e.g., Wertsch, 1985). An important challenge for the future will be to seek powerful syntheses of the views of those who emphasize analogy as (internal) mental model, those who focus on the structure of activity, and those who focus on mediational means, including discourse structures, to account for how initially quite context-specific developmental consequences come to have the degree of generality (however limited and uneven) that one observes as children grow older.

D.B., M.C., D. M.

References

Laboratory of Comparative Human Cognition (LCHC) (1983). Culture and cognitive develpment. In (P. Mussen, Ed.) *Handbook of child development, Vol. 1.* New York: Wiley.

Wertsch, J. (1985). *Vygotsky and social foundations of mind.* Cambridge: Harvard University press.

Learning Computer Languages and Concepts

Terezinha Nunes Carraher
Luciano de Lemos Meira
Universidade Federal de Pernambuco
Recife, Brazil

Computers have been introduced in education with several functions: as electronic books and blackboards, not representing any fundamental change; as a means of altering the social interactions in the classroom, with

0278-4351/89/1&2-2 $1.00 © LCHC

teacher and students solving problems together (Sutherland & Hoyles, 1986); as a device that allows students to use concepts which would perhaps be learned as goals in themselves but can, through computer work, be learned as tools (Carraher, Santos & Borba, 1986); as a significant way of teaching about logic and meta-cognition (Thompson, 1986); and as a goal in and of themselves, when learning to program (that is, learning computer languages) constitutes the purpose of the computer introduction in the classroom. This paper focuses on the introduction of computers in the classroom as a goal. Students' behavior when using a programming language—LOGO—will be analyzed in the same way that we analyze children's efforts in learning a natural language. It is generally accepted that when children learn a natural language, the acquisition process is not totally determined from the outside. There are important characteristics of the learner which determine the course of language development. When a child uses a word, for example, this use does not imply that the adult meaning was apprehended by the child. Children assign their own meanings to words when they first use them, and there may be a lengthy process of semantic development before the child can be said to share the adult meaning of many words. Papert (1980) already pointed out the similarity between natural language learning and learning LOGO; besides calling LOGO "TURTLE TALK," he asserts: "Since learning to control the Turtle is like learning to speak a language it mobilizes the child's expertise and pleasure in speaking" (p. 58).

This study describes the ways in which adolescents tried to attach meanings to the RIGHT and LEFT commands learning LOGO. It treats LOGO as a language in which meanings have to be constructed by the learners as they use the language. Learning LOGO is particularly interesting as analysis of the development of semantics for several reasons. **First**, there are no factors limiting production, in contrast to what happens in natural language acquisition where the child's ability to produce an utterance may prevent the child from saying a known word. Using the turn commands (RIGHT or LEFT plus some number) is not restricted in this study by the subject's inability to spell the turn commands or to write numbers using place value. Thus production data, which may be considered ambiguous in studies of natural language acquisition, can be assumed to represent reliable information on the semantic structure of the learners in LOGO use.

Second, Turtle Talk represents commands to be followed by the turtle. Contrary to what happens in natural language, where the listener may tacitly correct an utterance and try and respond to the meaning, not giving the learner explicit feedback on errors, the turtle is a literal interpreter: any command will be followed in the literal sense.

Third, the turn commands (RIGHT or LEFT) in LOGO are connected to the children's own situation, to the turtle behavior on the screen, and to the resulting drawing in such a complex way that learning LOGO is not like learning new words for old meanings; it is not like learning to say "gato" for cat or "cachorro" for dog. Learning LOGO requires that students organize meanings in particular ways, which may differ from those previously used in their daily world away from computers.

Turtle Talk, Turtle Behavior and Turtle Drawing

Before we describe the study, let us take a glance at LOGO and raise some questions about the difficulties in the acquisition of the turn commands with their semantic structures and relate Turtle Geometry to Children's Geometry as described by Piaget and his collaborators.

Turn commands in Turtle Talk are quite easy for any adult to produce. A turn command consists of a word, RIGHT or LEFT, which indicates which way the Turtle should turn, plus a number, which tells the Turtle how far to turn. RIGHT and LEFT, if already understood by the learners only become problematic when one looks at Turtle Behavior and Turtle Drawing as meanings of the commands. The Turtle in LOGO is a small triangle on a computer screen. A student working with LOGO is situated on a horizontal plane; the Turtle is situated on a vertical plane. This transposition in and of itself may be complex but other difficulties can be pointed out. A student writing commands has his/her right or left quite fixed, contrary to Turtle Behavior. The Turtle's right and left turns change in direction on the screen depending upon whether the Turtle is facing up or down, left or right at the moment. If the Turtle is pointing straight up and RIGHT command is issued, the Turtle turns towards the student's right. If the Turtle is oriented head down, the RIGHT command produces a left turn from the student's viewpoint. If the Turtle is oriented horizontally towards the student's right, the Turtle moves it head down after a RIGHT command. Finally, for a Turtle oriented horizontally towards the student's left the command RIGHT results in a movement that will straighten it up. Thus for the same command the Turtle behavior varies as a function of the Turtle's present orientation—a characteristic of LOGO which Papert treats as "intrinsic geometry" because of the reference to the Turtle's momentary position in determining the next movement.

The turn commands are always followed by a number "to say how much the Turtle should turn. An adult will quickly recognize these numbers as the measure of the turning angle in degrees" (Papert, 1980, p. 56). What meaning do children attribute to this number? When the student tells the Turtle to FORWARD 100, RIGHT 45, FORWARD 100, the Turtle draws on the screen an angle—in fact, two angles, an angle which is readily perceptible due to its closeness and measures 135 degrees, and another one, which we perceive by inverting the initial figure-ground relationship, and measures 225 degrees. Turtle Behavior and Turtle Drawing are thus related in ways which are not so simple. Putting the example in mathematical terms, *Turtle Behavior consists in turning an angle x which represents the supplement (that is 180 - x) of the angle seen after the Turtle has completed its Drawing.* If an adult quickly recognizes the number in RIGHT 45 as the measure of the turning angle in degrees, what meanings do students attribute to this number when they are attempting to draw something on a screen?

Turtle Geometry and Children's Geometry

Piaget, Inhelder and Szeminska (1960) described the development of the understanding of angles based on a task in which children were asked to draw two supplementary angles which they could study and measure as long as they wanted to but which they could not look at while producing their own drawing.

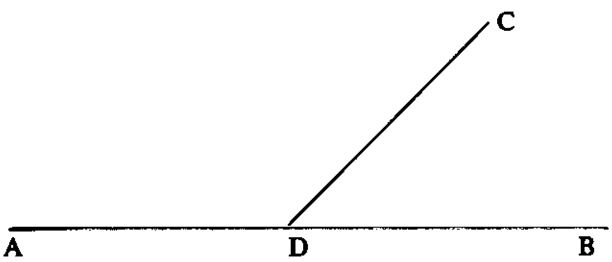


Figure 1. Angles to be reproduced in the Piagetian Task

They found that children's behavior could be interpreted as reflecting different strategies, which they organized into stages. In the first stage (4-5 years), children attempt to copy the drawing by visual estimates; no attempts at measuring are observed. In the second stage (6-7 years), children attempt to measure the lines in the figure to be reproduced (AB and DC) but their measurements are one-dimensional; no attempts at measuring the angles are observed and the slope of the line (DC) between the supplementary angles is not taken into account. In the third stage (7-8 years), children try to copy the slope by measuring the origin of DC and trying to maintain the slope of their ruler fixed as they move from the measured figure to their own drawing. In the third stage (8-9 years), children attempt to measure angular separation by adjusting the drawing in such a way as to make the distances AC and CB equal to the figure which is being copied. Although the children at this stage obtain the correct figure and use a two-dimensional operation in drawing, this measurement is not yet governed by one-to-one correspondence with two axes describing a plane. In stage four (about 10 years), children use a right-angle to measure angular separation, determining point C through the measurement of a perpendicular which connects C to the line AB in the figure. "In measuring an angle by dropping a perpendicular to the shorter of its arms, these children effectively integrate one-many correspondences within the framework of a rectangular coordinate system" (Piaget, Inhelder, & Szeminska, 1960, p. 184). Piaget, Inhelder, and Szeminska thus claim that, in their attempts to deal with angular measurement, children recreate Cartesian geometry by defining point C as a location in a plane determined by two perpendicular axes in a one-to-many system of correspondences.

We can compare Children's Geometry to the Turtle Geometry because Papert (1980) has made explicit what meanings he expects the turn commands to have from a broader perspective; that is, he has described what we could call "the intended semantic structure" of LOGO, which he calls the Turtle Geometry. The turn commands, he proposes, embody a geometry that differs from Euclidean and Cartesian geometry. When a child tells the Turtle to move a little and turn a little in order to draw a circle, the commands are said to refer "only to the difference between where the Turtle is now and where it shall momentarily be... There is no reference in this to any distant part of space outside of the path itself. The turtle sees the circle as it goes along, from within, as it were, and is blind to anything far away from it... For Euclid, the defining characteristic of a circle is the constant distance between points on the circle and a point, the center, that is not itself part of the circle. In Descartes' geometry, in this respect more like Euclid's than that of the Turtle, points are situated by their distance from something outside them, that is to say the perpendicular coordinate axes" (p. 67).

This very detailed description of the intended semantic structure of LOGO creates the opportunity for the analysis of an extremely interesting question in the acqui-

acquire the semantic structure as determined from the outside—that is, differential geometry, as intended by Papert—or is there a process of acquisition of meanings determined from within, as a consequence of the learners' own efforts to make sense of the language? With respect to the development of children's concepts of geometry, Piaget, Inhelder & Szeminska (1960) have proposed that there is a certain organization in development, which can be treated theoretically as the "organization from within." However, their studies pertain to a particular cultural milieu in which children are exposed to particular mathematical instruction and not exposed to LOGO. By looking at youngsters learning LOGO, it is possible to analyze whether a new cultural creation such as a computer language can result in changes in this developmental path.

The Study

This study describes students' strategies in designing programs using LOGO when they attempt to draw a target figure. It takes the viewpoint that they must understand the use of turn commands in order to draw particular target figures correctly from the outset, not in a trial-and-error fashion. LOGO enthusiasts seem to assume that children in fact learn LOGO by understanding intrinsic geometry (Papert, 1980; Kynigos, 1986). In this study, we ask: is it really so? The study describes different levels of development of the semantic structure underlying the production of turning commands by testing students who had different amount of experience with the language and different levels of previous instruction in school-geometry. We tested the students' competence by asking them to produce programs which would lead the turtle to draw particular figures—a production task. Other aspects of programming were also tested and students were given a comprehension task too. However, this report pertains to the production task only; details about the comprehension task and other aspects of programming can be found in Meira (1987).

Method

Subjects. Subjects were 32 Brazilian youngsters from a public school attached to the Universidade Federal de Pernambuco in Recife, Brazil, randomly selected from two groups of volunteers interested in learning LOGO. Half of the youngsters were 7th graders (approximate age 13 years) and had had very little training in geometry; half were attending their second year of high school, which is three grade levels up from the 7th grade (approximate age 15 years) and had learned some trigonometry. None of the subjects had had any previous experience with computers and none had their own computer at home. Although no prior testing of their competence in the Piagetian task of angle reproduction was carried out, it can be assumed from their age levels that most, if not all, would be able to carry out angle measurements by using a system of axes.

Design. In order to introduce more variance into the study so as to observe greater variation in strategies in programming, the students in each level of schooling were assigned to two levels of practice (15 or 30 hours) in LOGO prior to testing.

Procedure. During training, students worked in pairs with a non-directive instructor who allowed students to create their own projects and work freely at the editor during training hours. Between training sessions, students were encouraged to prepare projects that they would try to implement during sessions. Three different instructors worked in the training sessions with one instructor always assigned to the same students.

During testing, students worked alone with one experimenter who presented them with a drawing (Figure 2), and asked them to write programs for producing the figure away from the editor, turning to the editor only for debugging and at most five times. The figures were drawn on transparencies for overhead projectors which were attached to the computer screen so that students could compare their drawing with the target figure without difficulty. Students were interviewed according to the Piagetian clinical method; different questions were posed by the interviewer to students in an attempt to clarify the meaning of their answers but focusing especially on the turning commands and choice of programming units.

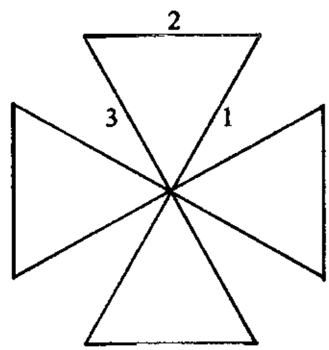


Figure 2. TRIANGLES to be copied in the LOGO Production Task

Paper, pencil, a ruler and two types of protractor (semi-circular and full-circle) were available to students during testing. Interviews were tape recorded and transcribed for analysis. Students' drawings and the interviewer's notes were coordinated with the protocols to help in building a more complete description of the students' behavior.

Results

Three types of strategy were identified in the students' attempts to attribute numerical values to the turn commands. Since these strategies are strongly correlated with the ability to achieve a correct drawing, they will be treated here as levels of competence in LOGO, describing different types of semantic structure for the turn commands. No assumptions about the students' cognitive development are made on the basis of this analysis; we assume that variation here refers to the development of competence on the acquisition of a new language.

Students in Level I, the lowest level of competence, seemed to treat the numbers attached to turn commands as *labels for lines or angles obtained after turtle drawing* has been completed. Turtle behavior and turtle drawing do not seem to be distinguished by these students. The following protocol illustrates this approach.

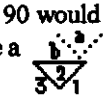
Flavia: (trying to do line 2 in the figure) I will have to use 90.
Interviewer: Why?
Flavia: Because 45 is like this (shows tilted line) and 90 is like this (shows horizontal line).

Flavia's behavior illustrates line-labelling; she treats the numbers as labels for the lines obtained when the turtle is through with its drawing and does not take into account the fact that the effect of a 45 or 90 turn will depend upon the momentary position of the turtle. Angle labelling is illustrated in the protocol below.

João: (trying the same turn, after using RIGHT 45 for line 1) LT 120.
Interviewer: How do you know that it is 120 at the tip?
João: It's the angle here, isn't it? It's more than 90, I suppose that it is 120 because it is the most used angle.
Interviewer: Most used how?
João: It's the closest one.
Interviewer: Do you think it could be 130?
João: (...) I think it (120) will work?

Although the student chooses the right value for the angle in the *actual figure*, he does so for the wrong reason since his initial choice of turn command, RT 45, followed by LT 120, will not yield the desired result, a horizontal line. The angle 120 chosen by João is the supplement of 60, the internal angle of an equilateral triangle; it is in fact "the most used angle" when students are drawing triangles with LOGO.

Students working at Level II show what can be seen as a transitional level of competence between Level I and Level III. Contrary to Level I students, they clearly distinguish between Turtle behavior and Turtle drawing. They use *labelling of angles but refer to Turtle behavior and not Turtle drawing.* On some occasions, they attempt *an additive composition of angles*, but these angles are chosen through visual estimates and are not yet described in terms of their deviations from an imaginary systems of axes, as they will be in Level III. As a result of visual estimation, Level II students may add the wrong measures of angles. The protocols below illustrate students' strategies at this level of competence.

Marcel: uses RT 45 to obtain line 1 in the figure and LT 90 to obtain line 2; he then erases 90 and writes down 110.
Experimenter: You had written 90, hadn't you?
Marcel: Yes, 90 would be like this (shows the right angle a 

) but here it would be more.
Experimenter: O.K., but why is it 110?
Marcel: I thought this difference (b) would be 20, it is a small difference, 90 would leave this small difference... This part after the 90 turn is small and the turn of the Turtle with a 20 is small.

Marcel uses the initial turn of 45 perhaps labelling a tilted line as 45 and a horizontal line as 90. Then he reexamines his choices and correctly realizes that a turn of 90 to the left when the Turtle starts from a position titled to the right will not bring it to the horizontal position. He correctly imagines that a 90 turn will bring the Turtle to the position obtained after turning a right angle. He clearly dissociates here Turtle Behavior and Turtle Drawing thereby making a correct prediction about turtle behavior. However, he does not use the deviations from the system of axes to carry out the additive composition of angles; he estimates the difference between the predicted position of the turtle after a 90 turn and the intended position and adds the difference to the 90 angle.

Finally, at Level III students clearly distinguish Turtle Behavior and Turtle Drawing and no longer work solely by visual estimates. Their choice of *turning commands reflects an additive composition of angles which are described with reference to a system of imaginary axes*, deviations from which are treated as quantified angles. The protocol below illustrates this conduct.

Experimenter: How did you find 120 for this turn?

Vania: Because I saw that this one (used to obtain line 1) was 30, plus 90. . . 30 to bring it back to make it upright plus 90, which makes it become horizontal, that it 120.

Experimenter: Did you use the protractor?

Vania: I looked at it (but doesn't place it on the figure to obtain the measurement).

Experimenter: Will you use 30 and 90 for all the other tips of the triangles?

Vania: Yes, but here (to obtain line 3) you go 90 first and then 30, that will give you 120 again...

Experimenter: How can it be the opposite of the other one?

Vania: Because you have to know where the head of the turtle is. In he first corner, it was like this (shows the turtle's position after the first turn), so you go 30 to make it upright and then 90 to make it horizontal. In this second corner it starts from his position (shows a horizontal line).

The frequency of students per level of competence as a function of grade level and amount of prior training is shown in Table 1. The distribution of high school students with 30 hours of prior training differs significantly from a random distribution according to the Kolmogorov-Smirnov test (D = .542; p = .01); all other distributions are essentially random. This result indicates that older students are able to acquire more quickly a good semantic structure for dealing with turn commands in LOGO.

Table 1

Frequency of students in each level of competence in turn commands by grade level and number of hours of LOGO instruction:

	7th Graders	High-School		
	15 h.	30 h.	15 h.	30 h.
Levels of competence				
Level I	2	4	3	1
Level II	2	1	2	0
Level III	4	3	3	7

Discussion

The analysis of levels of competence in choosing turn commands clearly shows that the semantic structure implicit in the strategies of the more competent LOGO users in our study is rather different from Papert's expectations. It is clearly coherent with Cartesian geometry—angles are measured as deviations from an imaginary system of axes—and thus coherent with Piaget's and his collaborators description of children's geometry. It is possible that with further training children would eventually acquire a semantic structure for dealing with turn commands more coherent with Turtle Geometry. We have no evidence for this acquisition within the first 30 hours of training. It is also possible that non-directive training is not the best way of leading students to the acquisition of Turtle Geometry; students may need to be explicitly told to put themselves in the Turtle's place in order to acquire the body syntonic geometry of the Turtle. If they learn LOGO by simply looking at Turtle Drawing on the computer screen, they may be placed in a learning situation which does not favor body syntonicity.

In summary, we found no evidence for the acquisition of a semantic structure implicit in turn commands directed from the outside by the intended semantic structure of the language. The evidence clearly indicates that the changes in the semantic structure of the turn commands follow the same path described by Piaget and his co-workers in children understanding geometry prior to LOGO and Turtle Talk.

Note

[1]Papert (1980, p. 100) presents a program by a student which can be clearly identified as an example of line labelling; the student chooses a turn of 120 every time he wants to draw a line with a particular inclination despite the fact that his first turn started with the turtle facing up while in all other turns the turtle started the turn from positions in which it was tilted.

References

Carraher, D. W., Santos, A. L., & Borba, P. (1986). *Sherlock Exploring English.* Computer Program, Unpublished Manual. Universidade Federal de Pernambuco.

Hoyles, C., & Sutherland, R. (1986). In C. Hoyles, R. Noss, & R. Sutherland (Eds.), *Proceedings of the Second International Conference for Logo and Mathematics Education.* London: University of London Institute of Education, 54-63.

Kynigos, C. (1986). Constructing a conceptual pathway from intrinsic to extrinsic geometry. In C. Hoyles, R. Noss, & R. Sutherland (Eds.), *Proceedings of the Second International Conference for Logo and Mathematics Education.*. London: University of London Institute of Education, 64-71.

Meira, L. de L. (1987). *Geometrias em acao na programacao em LOGO.* Masters' Thesis, Universidade Federal de Pernambuco.

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas.* New York: Basic Books.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry.* New York: Basic Books.

Sutherland, R., & Hoyles, C. (1986). Mathematics and LOGO—A development in our intervention strategies. In C. Hoyles, R. Noss, & R. Sutherland (Eds.), *Proceedings of the Second International Conference for Logo and Mathematics Education* 262-269. London: University of London Institute of Education.

Thompson, P. W. (1986). Logo as a context for thinking about thinking. In C. Hoyles, R. Noss, & R. Sutherland (Eds.), *Proceedings of the Second International Conference for Logo and Mathematics Education.* London: University of London Institute of Education, 209-215.

The Role of the Personal in Intellectual Development

Herbert P. Ginsburg
Teachers College, Columbia University, USA

The essence of intellectual activity is usually considered to involve thinking, rationality, reason. Perhaps the paradigm case is "logical" thinking of the sort described by Piaget, as when the child reasons according to principles of transitivity, or engages in the hypothetico-deductive logic characteristic of formal operational thought. Typically we see this kind of thinking as "cold" and lacking in affect. One explanation for this is that Piaget focused on intellectual "competence"— the nature of underlying intellectual structures, unaffected by performance variables like motivation, fatigue, environmental obstacles, and the like. Piaget was concerned not with what people *ordinarily* do, but with what they are *capable* of doing under favorable environmental conditions.

0278-4351/89/1&2-8 $1.00 © LCHC

This is a valuable perspective, sometimes unappreciated, particularly in American psychology. But to understand behavior more fully, it is also necessary to assume another perspective, focusing on *performance*, not *competence* alone. If the aim is to understand what people ordinarily do and why they do it, then one must widen the scope of inquiry to consider issues of personality, motivation, and the like. Taking this perspective requires an examination not only of the underlying cognitive structures, but of the "learning life," of the way in which the individual conducts his or her intellectual activities in the real world.

Furthermore, understanding the learning life requires a depth psychology account of intellectual development, not merely a cognitive one. Contrary to Piaget, intellectual development is not primarily the development of logico-mathematical structures motivated by an equilibration process. Contrary to Vygotsky, intellectual development is not primarily the socialization of advanced forms of intellect. And contrary to the information-processing theorists, intellectual development is not primarily the acquisition of procedural and declarative knowledge. Instead, intellectual development is intimately bound up with the development of the personal—identity, defense mechanisms, motives, emotional attachments, and the like. Only by developing a depth psychology of intellectual development, difficult and ambitious as the task may be, can we understand the complexity of intellectual performance in the real world. And perhaps such a framework will help us to develop educational and clinical practices designed to avoid squandering children's enormous intellectual potential.

Method

My argument is buttressed with examples drawn from our recent research, as well as anecdotal accounts. This research, conducted over the past several years, involves case studies of children and adults, most with learning problems. Several groups of individuals were seen:

A. College students at a prestigious university who were doing at least adequate work and were willing to talk about their learning experiences.

B. College students at a large urban university (seen in collaboration with Professor Ned Mueller) who felt that they were experiencing difficulties in the academic area.

C. Students at a small college who seemed to be experiencing a very traditional form of education.

D. Graduate students at a large urban university who considered themselves to suffer from "math anxiety" (Ginsburg & Asmussen, in press).

E. Asian-American college students at a prestigious university who exhibited high levels of academic achievement (Mordkowitz & Ginsburg, 1987).

F. Children in a mental health clinic (seen with Prof. Mueller) and children in ordinary schools, all with learning problems of varying degrees of severity.

These individuals were generally interviewed in an extensive manner—for at least an hour, often two or three hours, and in some cases for six hours. The interviews were usually guided by a flexible protocol which left a good deal of room for variation in response to the individual. In several cases, a brief projective measure, The MUG (Mueller & Ginsburg, in preparation), was administered. Also, in some cases, both children and adults, we observed problem-solving activity (in mathematics) directly. The clinic children seen with Mueller were also given several standard tests (like the WISC). All of the sessions with children and adults were tape recorded and transcribed, some verbatim and some in summary. These "data" were then interpreted in what I hope are creative ways and lead to the propositions forming the body of this paper. It concludes with a discussion of the implications of this view for fostering intellectual growth.

The Nature and Personality of Intellectual Activity

One aim of the interviews was to obtain some insight into the nature of the individual's learning life. What is it like to be a thinker and learner in the everyday world? Our results show first that everyday intellect is suffused with emotion.

Emotion

Cognition is "cold and affectless" mainly when we abstract it from the living reality. My first proposition is that *in reality, emotion is part of intellect.* Of course, many writers agree that emotion, motives, and the like can *influence* intellectual activity, as when a child fails in school because he is poorly motivated or because anxiety overwhelms his thought. But I mean to go further, claiming that emotion is not a separate factor which influences intellect. Rather, it is impossible to separate emotion from cognition; intellectual performance is suffused with both.

For example, Butch (described in Ginsburg, 1989), a third grade child, exhibited depression in his mathematical problem-solving. Everything he did in arithmetic was listless, slow, and expressed helplessness and incompetence. His work was not just wrong; it expressed misery. For Arnold, another third grader with learning problems (Ginsburg & Allardice, 1984), the emotion was overwhelming and even found bodily expression. Arnold said: "I usually hate math because it hurts my fingers ... it makes my fingers feel frail. It feels like I can't write ... It hurts. I feel like I'm going to snap the pencil in half." A rather dramatic woman of about 50, Lucy, a graduate student, claimed that she "almost fainted, almost died" when she had to deal with mathematics, and became "nauseous, tense, with a buzzing in my ears." At one point, during a problem solving session, she insisted in terminating her work lest she throw up in my office, a possibility neither she nor I wished to explore. Clearly for Lucy, emotion is a part, a very unpleasant part, of her intellectual work in this area, and her mathematical thinking did not operate in an emotional void. Emotion and cognition were blended in an ongoing activity. And for many individuals, the emotion of intellect is unpleasant, negative, hostile, and painful.

For others intellectual performance involves peak experiences, and even a "love" relationship. Mary, a student at a prestigious college, reported that she "loves" reading. For her, that activity is filled with a glow, and makes her feel exuberant. Paul, another student at that college, felt the same way about writing. He reported that the mere act of putting words together in a logical and elegant fashion entails an ecstatic experience. Arthur, also a student at that college, was asked why he is majoring in physics. He replied, in a serious manner: "It's something I was born with... I feel like it's great, I feel like I'm fulfilling the master plan for Arthur Rivers. I mean he's supposed to go to college and learn physics... It just feels right. I mean I love studying it. I love studying other things, but this is the thing I love studying more than other things... It feels good to know the stuff, to learn about it, I'm fascinated by a lot of it and so it's something that I kind of eat up, you know, like good food when I'm hungry."

Traits of Intellect

Not only is intellect suffused with emotion, in everyday life, we tend to describe it in terms of various "traits." For example, in an informal study of letters of recommendation I found that professional psychologists do not describe graduate students in terms of IQ, formal operations, or information processing techniques, but in terms of traits (often in situational context) like independence of mind, risk taking, creativity, originality, flexibility of thought, intense motivation, and the like. I have also found that clinicians use similar concepts to discuss their learning disabled clients (although the opposite dimensions are used). Further, use of similar trait concepts is a key part of clinicians' interpretation of subscales of the WISC (see Rapoport, Gill & Shafer, 1976).

The traits of intellect usually fall into several areas, among them cognitive style and motivation. *Cognitive style* is an old and complex topic in psychology but currently an unpopular one. Like Messick (1984), we use the notion of style to refer to "habits of thought," to ways of thinking that make the individual unique and constitute his "personal signature" (Goodman, 1978). In our observations in the clinic and in everyday life, we find that some styles seem to be crucial, producing beneficial or deleterious effects depending on the context; styles are not uniformly effective or ineffective.

An example of a style harmful to problem-solving in most contexts involves a strong rigidity of thought, as displayed by several graduate students in a study of "math anxiety" (Ginsburg & Asmussen, in press). Their tendency was to approach problems in one way only, usually relying very heavily on rules and formulas thought to be appropriate. For example, one subject was given a problem in which the goal was to determine whether a sum of two fractions was correct (1/32 + 3/16 = 1 1/8). Lucy engaged in extensive calculations, converting to common denominators and getting into a good deal of trouble along the way. A simpler solution would have been to look at the relative magnitudes of the fractions, noticing that since they were both so small, they could not possible add up to a sum larger than 1. But Lucy, like others, was stuck on the limited approach; her rigidity prevented her from using her "common sense."

While we tend to have little confidence in style tests, partly because of various psychometric problems (although see Zimiles, 1986, for a more general critique), we are quite committed to the use of style concepts in everyday life. We are all expert at describing styles, for example, those of colleagues and students, and our repertoire of style terms is usually considerable. In this case we should trust our naive psychology: clearly intellectual performance must be described not only in terms of information processing, but cognitive style as well.

Motivational terms are also frequently employed to describe intellectual activity. We see people as deeply committed to their work, engaged, intense, workacholic, and the like. A number of psychologists have recently focussed on issues of academic motivation, investigating the influence of non-cognitive factors on intellectual activity (for example the work of Ames & Ames, 1984). Thus individuals whose tendency is to ruminate on failure tend to exhibit lower levels of achievement than do those who engage in active attempts to solve problems. Those who wish to look good in the eyes of others tend to take fewer risks in learning and to avoid getting wrong answers. Clearly motivational characteristics like these can be central to intellectual performance.

It is also important to describe the learning life in terms of *concepts of the subject matter* and the *self*. For example, Lucy maintained the explicit belief that mathematics *must* be done quickly. "I want to get it done fast and I don't like to show the guts of it, the messiness of the page. I don't want to work through it. I want to have the answer come from the sky. Everyone else gets it quickly." Lucy's theory of mathematics learning is widely shared. Many students (and teachers and parents) believe that mathematics is a subject in which one attempts to get right answers as quickly as possible.

In another study (Kaplan, Burgess, & Ginsburg, in press) we showed that some children believe that the essence of mathematics is pleasing the teacher. This is their primary cognitive representations of mathematics. Talk about principles and strategies is quite beside the point.

Many of our subjects did not believe or understand that mathematics is a way of thinking, and that it is quite acceptable to struggle through to a solution which may be only one of several possible. And clearly this belief then influences the way in which problem solving is conducted. Many children exhibit no thought in doing mathematics—they take wild guesses and act stupidly—precisely because they believe (often because they have been *taught*) that mathematics requires quick answers, or that one must get the answer the teacher has in mind, and that as a corollary, thinking is cheating.

Self-concept or *theory of the self as learner* also influences learning. We all know, and our evidence confirms, that many children do not believe they are capable of learning, at least in certain areas. Sometimes, as a result, they avoid learning situations, they act the fool, they do not struggle. Many elementary school teachers hold such views of themselves in regard to mathematics learning. That is one reason why they avoid teaching the subject at all costs and why children learn it so poorly.

A more productive view of self as learner of mathematics might involve discovering, for example:

What one's capabilities are—What one can and can not learn.
What one feels about mathematics.
How long it takes one to learn something.
That one can learn even if initially confused.
That it is acceptable if one does not know everything.
That one can ask for help.

In other words, the concept of self as learner is not global. In involves theories, facts, speculations, and feelings about one's intellectual activities. It is a series of emotional propositions about oneself that seem to play an important role in learning and motivation.

The Personality of Intellect

Furthermore, *the constellation or configuration of these traits can be said to describe the "personality" of one's intellectual life*—that is, what separates the person from others and reveals his or her "distinctiveness." Normally we speak of personality as if it were limited to the constellation of interpersonal traits (aggressiveness, dependency, etc.) operating in the social world. But such a restriction is not necessary. Intellect has a personality too, and like one's social personality it may be rooted in deeper structures.

Shapiro (1965) has given a fascinating description of several such personality types, which he refers to as "neurotic styles" of cognition. These include, for example, the obsessive thinker whose thoroughness leads to the creation of unnecessary puzzles to be solved. True, Shapiro's subjects were "patients" in analysis, but they seem quite similar to "normals" with whom we are all familiar.

The description of a personality of intellect in terms of motivation, style, concept of self as learner, and the like is valuable but somewhat unsatisfying since it is silent with respect to origins and dynamics. Personality descriptions like these are superficial in the sense that they deal with what is on the surface. We see that rigidity clearly characterizes Lucy's behavior. But we must go further and ask: Why did she develop such an orientation and why does she maintain it? What are the origins of the rigidity, of the sense of identity? Why does this complex system operate as it does? Answering questions like these requires consideration of dynamics and development—a depth psychology. Understanding what underlies the personality of cognition requires looking under the surface, just as we might try to interpret a dream, to examine deeper processes.

The Dynamics of Intellect

Psychoanalysts have made the most important contributions to understanding this relatively unexplored subject. In this view, *one may understand the personality of intellect—just like the social personality—as the product of underlying processes like the mechanisms of defense, temperamental factors, and the like.* Thus Shapiro describes obsessive thinking as one aspect of the obsessive character. These traits of thinking, or others such as dependency of thought, are in part the results of defenses, those deep-seated, unconscious ways of coping that are established early in life, fundamental to character, and often very difficult to change. I speculate that some styles are less changeable than others precisely because the former are more deeply embedded in personality structure than the latter.

Another example of dynamic explanation is Anna Freud's (1946) notion of the "restriction of the ego," a most useful concept in understanding learning difficulties. For some children, openness to the world and to the kind of learning that can result is so painful that they pull in the ego, protect it, and shelter themselves in ignorance, or in rote, mechanical learning. The result is children with a resistance to experience, with little curiosity, few intellectual interests. These deep-seated forms of motivation can over-ride the equilibration process described by Piaget and can perhaps explain such surface characteristics as Dweck's notion of performance orientation.

Development

The emotion of intellect often begins early in life. Several of our subjects report that intellectual emotions get established early in life. Some children say that they "always hated" math, as long as they can remember. My observation is that in the first few years of school, most children are interested in and feel good about all forms of learning. But, within a short period of time, many children develop strong antipathies toward the learning of particular subjects and, by the third or fourth grades, are essentially lost to the educational system. If there is any critical period for the establishment of negative feelings toward school learning, it must be in the early years of school.

Falling in love with learning also gets established early in life. This is a kind of bonding process, a kind of object relations in the psychoanalytic sense, that involves particular subject matter. Paul's infatuation with writing developed, as he remembers it, somewhere around the 8th grade, and has persisted. Arthur claims that he was "born" with his interest in physics. While this is an exaggeration, the interest seems to have begun in elementary school. Seymour Papert (1980), mathematician, psychologist, and creator of the first major computer language for children, reports that he "fell in love" with gears at the age of four and that this served as the prototype for his interest in mathematics thereafter. Mary reports that one of her central interests, reading, developed in the first few grades: "My favorite sort of school memories are about reading... It was very exciting... It enabled me to sort of... hear about the lives I didn't have. It told me what other people's lives were like."

These intellectual relationships may be tied to significant personal experience. Mary reported that one of her major interests involved understanding and helping others. "One of the reasons that I'm interested in counseling is that... I grew up in an alcoholic home. It was extremely difficult. It has determined a lot of my interests in the ways in which I want to help people and also I'm fascinated by alcoholics. I'm also interested in literature because I feel there are certain authors who write about how individuals come to feel certain ways and make certain choices and by studying that it makes more clear my own recent behavior... Having had painful experiences growing up with my mother's alcoholism I am able to have compassion with other people's pain and I am naturally drawn to helping people to try to learn certain things about themselves." In brief, intellectual relationships are often bonded early in life, and may be tied to personal experiences.

What is the role of significant others in the development of intellect? Some writers have described a kind of instructional process (e.g., Hess & Shipman, 1965), and others have postulated a more direct and sensitive process of "scaffolding" (e.g., Greenfield, 1984 and Rogoff & Gardner, 1984). But in my view, it is worth considering other processes like *identification* and *expectation*.

Intellectual interests are often mediated by a process of identification with significant others. For Mary, the most significant person was her mother. "[Throughout school] the most influential relationship was probably with my mother. She's the person who would help me get books out of the library and she always read my papers and she helped me sort of form my taste." But teachers were also important for Mary. In particular, she remembers a second grade teacher. "I remember her because she helped me to read alone and she responded to the fact that I really liked to read. It was always important to me to know that they cared about reading too. I would identify very strongly with a teacher if they really cared about something." So for Mary, it was useful for the significant other to help her with the technical aspects of reading; but it was more important that the other be a good model and resonate to her intellectual passions. Similarly, Paul reports that his interest in writing was mediated by a teacher who not only taught the mechanics of grammar but *lived* a passion for the subject. Teaching cannot be reduced to instruction; the living commitment and moral example are central. This of course is an ancient principle of Zen Buddhism, in which the master does not teach but *is*.

Intellectual development, perhaps especially in the area of academic learning, can be strongly influenced by general parental pressure and expectation. Interviews of Asian-American students concerning the ways in which their parents influenced their academic achievement (Mordkowitz & Ginsburg, 1987) yield several striking phenomena. Asian-American parents expect their children to do well in school, place a high value on education, put pressure on their children to do well, and arrange living conditions so that the children can do well. The parents may or may not help the children with homework. Indeed, some of the parents may be so poorly educated themselves or so unskilled in English that they may not have the ability to teach. This phenomenon is similar to the Jewish immigrants in the US in the early 1900's. While many were unschooled, they had a respect for learning and expected their children to do well in school. Similarly, the Asian-American parents often force the children to do homework and achieve good grades. They release the children from household responsibilities so that they will have the time to study. The message is: I'm sacrificing for you so that you can learn. The parental influence, then, operates not through teaching or scaffolding but through general pressure and control, perhaps mediated through carefully nurtured guilt.

What promotes development? A kind of self-directed process seems to be involved, but it is not equilibration in the Piagetian sense. *Intellectual growth often involves finding a way to connect significant personal concerns with school learning.* Sometimes Mary could muster little enthusiasm for school work. She found it hard to concentrate on her studies and did slipshod work. Why? Mary felt that many of her difficulties in academic work stem from its isolation from her personal concerns. "Academic life is definitely separate from your personal life, separate from your family life... but you can't separate it entirely... and when you ignore that a lot of problems come up... academics is a very personal thing... it is based not only on intellectual things but on emotional things." As her confused presentation indicates, Mary was obviously struggling with these issues. For her the central learning problem was to find a way to relate her studies to the meaningful issues of her life. If this cannot be done, she suffers from lack of interest and motivation.

Intellectual growth often involves working through issues of identity. Woody is a college student who has experienced considerable difficulty in his studies. He came from a family in which his father is a Professor of Mathematics and his older brother an excellent student in chemistry and computers. Woody felt strong pressure to excel in academic work and to go on to a career in mathematics. He sees this as the fatherly, brotherly thing to do. Woody gives the impression of being effeminate. When he realized that the interview was to be videotaped, he said, in a half-joking fashion, "If we're going to be on camera, I'll fix my hair." He consistently used fancified language, referring for example to a picture of a brutish looking person as "this gentleman" and referring to his own depression as "early morning awakening psychomotor retardation." In response to one of the pictures of the MUG projective test—a person sitting with some books in a cozy living room—he says: "This young man is studying organic chemistry and is in a very comfortable home and he's not too pleased studying because he sort of feels like he'd rather be over here eating this banana... there's attractive fruit over here. He eats the fruit and smells the flowers and you really wonder why he's taking such hard science courses." Like the young man in his story, Woody feels that he must take hard science courses because that is the masculine thing to do, as determined by his father's and brother's example. But at a deeper level, Woody's preferred identity is feminine. He would rather eat the fruit and smell the flowers. Our interpretation is that Woody's academic difficulties are related to his identity conflict. To resolve the conflict he needs to find what *he* defines as a feminine outlet for his intellectual concerns.

Conclusions

Our exploratory research points to a few general conclusions. First, that the learning life is clearly "hot": it is *bound up with* emotions, beliefs, styles, motives, and identity. We use the ambiguous phrase *bound up with* because the relations among these "factors" or "processes" and intellectual activity are complex. In one sense, some of the factors are *parts* of intellectual activity in the real world. Mathematical thinking, for example, can involve more than strategies and concepts. It can also be fearful, dependent, and persistent. We might even say that thinking has a "personality" which needs to be described and explained.

In this view, we need to broaden our conception of what intellectual activity is all about. It is more than strategies, knowledge, and procedures. Instead, intellectual activity *is* in significant measure the operation of beliefs, feelings, motives, and the like. As Dewey (1933) pointed out many years ago, intellectual activity does not exist in isolation from other aspects of the person.

Second, to understand the complex nature of hot cognition, or the personality of cognition, we need to consider both dynamics and development. Thus we need to understand how the observed intellectual activity, style, feeling, and motivation operate within the context of the individual's identity and general personality structure.

Developing such a theory is an important task for the future. We suspect that one important resource for such a theory is the "ego psychology" stemming from the psychoanalytic tradition, as exemplified by the work of such writers as Shapiro (1965). In this tradition, cognitive functioning, in both the "normal" and pathological, is seen as but one aspect of ego development, which in turn must be understood in the context of the dynamics of personality, including such matters as the ways in which cognitive styles emerge from the functioning of the defense mechanisms. In other words, the personality of cognition must be understood in terms of the dynamics of personality.

Third, it must also be understood in the context of development. Hot cognition has a history, and this history may often involve relations with the parents or significant others. Understanding of thinking—emotion—motivation requires a truly developmental approach which attempts to elucidate the meaning of current styles, emotions, and motives in terms of attempts to cope with early experiences of various types. Thus, a "performance orientation" may be understood by reference to a dependency relation with a parent; an ahistorical approach sheds little light on the meaning and origins of the motivation.

Fourth, our approach to understanding intellectual activity in the context of the personal suggests a somewhat different approach to education. In our view, education is more than acquiring information or cognitive skills or getting good grades on tests. For many students, education is not just the cognitive activity we thought it was. It is something more personal, more deep. Most crucially, it involves finding personal meaning in what is taught in school. One crucial aspect of education involves the integration of the formal into the personal. This in turn is intimately bound up with the development of identity, of defense mechanisms, of emotional attachments. Real learning is not just mastering skills; it is at least in part the process of creating meaning by connecting what is taught to what the individual sees as important.

One way of interpreting this kind of integration is in terms of Piaget's and Vygotsky's theories. Piaget (1952) pointed out that assimilation is the "prime fact of mental life." By this he meant that we always interpret the new data of experience in terms of what we already know, in terms of existing structures. For many students, the new data are the subjects taught in school and the most important existing structures are their personal identities. (Thus Mary had to integrate school work with her concern with understanding and relating to others.) Vygotsky (1962) pointed out that the chief task for education is to integrate spontaneous and schooled knowledge, to blend the personal and the social. For many students, the integration must involve not only schooled knowledge with informal knowledge (for example, written and mental arithmetic). It must also involve connecting schooled knowledge with intimate personal concerns. If that kind of integration is not achieved then many individuals find that schooling, however successful it may appear by conventional criteria, is merely "academic"—that is unimportant, personally meaningless, and irrelevant.

The study of the learning life thus points the way to more meaningful education. Certainly the educational process cannot resolve many of the emotional conflicts which inhibit children's learning. Nor can the educational process modify character. But there is much that can be done in school to make education personally meaningful. If knowledge is also emotion and motive, and if knowledge develops in the context of the dynamic personality, then a concern with cognition and instruction is not enough: education must involve more than the transmission and even reinvention of knowledge. Educators need to learn to foster the integration of the formal and the personal. They need to develop techniques to help students understand their learning styles, their feelings about learning, their identities as learners. Traditional psychotherapies generally fail to do this. They focus primarily on interpersonal relations. Traditional counseling approaches also fail to focus on learning related issues. So we need new approaches to help students find meaning in formal education; this will go at least part of the way towards liberating their potential for intellectual growth.

Note

[1]I wish to express my appreciation to the Spencer Foundation for its generous support of the preparation of this paper.

References

Ames, R. E., & Ames, C. (Eds.). (1984). *Research on motivation in education.* Vol. 1. Student Motivation. Orlando, FL: Academic Press.

Dewey, J. (1933). *How we think* Chicago: Gateway.

Freud, A. (1946). *The ego and the mechanisms of defense.* New York: International Universities Press.

Ginsburg, H. P. (1989). *Children's arithmetic* (Second Edition). Austin, TX: Pro Ed.

Ginsburg, H. P., & Allardice, B. S. (1984). Children's difficulties with school mathematics. In B. Rogoff & J. Lave (Eds.), *Everyday cognition* (pp. 194-219). Cambridge, MA: Harvard University Press.

Ginsburg, H. P., & Asmussen, K. A. (in press). Hot mathematics. In G. Saxe (Ed.), *New directions in child development* San Francisco: Jossey Bass.

Goodman, N. (1978). *Ways of world making.* Indianapolis: Hackett Publishing.

Greenfield, P. M. (1984). A theory of the teacher in the learning activities of everyday life. In B. Rogoff & J. Lave (Eds.), *Everyday cognition* (pp. 117-138). Cambridge, MA: Harvard University Press.

Hess, R. D., & Shipman, V. (1965). Early experience and socialization of cognitive modes in children. *Child Development, 36,* 869-886.

Kaplan, R. G., Burgess, P., & Ginsburg, H. P. (in press). Children's mathematical representations are not (always) mathematical. *Genetic Epistemologist.*

Messick, S. (1984). The nature of cognitive styles: Problems and promise in educational practice. *Educational Psychologist, 19,* 59-74.

Mordkowitz, E. R., & Ginsburg, H. P. (1987). Early academic socialization of successful Asian-American college students. *Quarterly Newsletter of the Laboratory of Comparative Human Cognition, 9* (2), 85-91.

Mueller, E. C., & Ginsburg, H. P. (in preparation). *The MUG projective test.*

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas.* New York: Basic Books.

Piaget, J. (1952). *The origins of intelligence in children.* New York: International Universities Press.

Rapoport, D., Gill, M. M., & Shafer, R. (1976). *Diagnostic psychological testing* (rev. ed.). New York: International Universities Press.

Rogoff, B., Gardner, W. (1984). Adult guidance of cognitive development. In B. Rogoff, & J. Lave (Eds.), *Everyday cognition* (pp. 95-116). Cambridge, MA: Harvard University Press.

Shapiro, D. (1965). *Neurotic styles.* New York: Basic Books.

Vygotsky, L. S. (1962). *Thought and language.* Cambridge, MA: MIT Press.

Zimiles, H. (1986). Guiding the study of cognition to a framework of greater complexity. In D. J. Bearison & H. Zimiles (Eds.), *Thought and emotion: Developmental perspectives.* Hillsdale, NJ: Lawrence Earlbaum Associates.

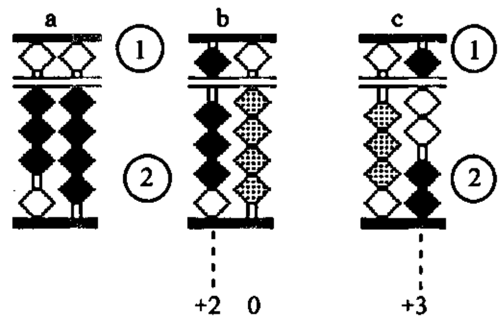
Two Microworlds of Computation: How are they Related?

Giyoo Hatano
Dokkyo University, Japan

A large number of Japanese children learn abacus operation in addition to paper-and-pencil calculation. Abacus operation, still used daily at small shops, is learned either without systematic instruction or at specialized kind of private school (called an abacus *juku*), while paper-and-pencil calculation is taught in regular math classes. From a cognitive point of view, these two sets of skills are highly similar: they share the goal (finding the answer) and component subskills (e.g., use of basic number facts, borrowing from the next left). They are different primarily in two aspects. (1) While in paper-and-pencil calculation a figure written in the answer column is not modifiable, a resultant number on an abacus can easily be changed, which allows abacus operation to proceed from the largest to the smallest (units) figures, whilepaper-and-pencil calculation goes in the opposite direction. (2) Since the Japanese abacus has a bead worth 5 in the upper section (instead of nine 1-beads) in each column, carrying and borrowing procedures on the abacus are a little more complicated than those with paper and pencil. See Fig. 1 for an example.

0278-4351/89/1&2-15 $1.00 © LCHC

Figure 1. Addition with an abacus, 34 + 23 = 57



a. Enter 34.
b. Enter the 5-bead and remove three 1-beads in the tens column, since the 2 in the figure 23 cannot be added by using 1-beads.
c. Enter 5 asnd remove 2, since the 3 in 23 cannot be added by using 1-beads.

From a socio-cultural point of view, however, the two sets of skills are segregated almost completely. Students are not allowed to use an abacus for solving problems in regular math classes (except when they learn how to operate it in the 3rd grade). Paper-and-pencil calculation is almost never done at an abacus *juku.* Thus they often regard the sets of skills as belonging to two separate microworlds, in only one of which each set can legitimately be used.

Therefore, two contrasting predictions can be derived about transfer from abacus to written calculation. One may expect there to be "analogical" transfer from one set of skills to the other: more specifically, one may expect that if students are good at abacus operation, they can use their knowledge about how to run the operation when they work on paper-and-pencil calculation. Quite to the contrary, one may equally well expect these sets of skills to be completely independent, i.e., that practice in abacus operation has no effects on calculation with paper and pencil.

In this short article I would like to claim that neither prediction is right. First, I will show that it was very difficult for children to improve written calculation by using their knowledge of abacus operation. Second, I will demonstrate that these sets of skills are not independent: abacus operation influences written calculation, probably through enhanced proficiency in shared component subskills of basic computation. Finally, characterizing abacus operation as a form of non-school mathematics that is radically different from "street mathematics" observed by Carraher, Carraher, and Schliemann (1985), I will derive a few implications for mathematics instruction.

Mapping Instruction Does not Work

Amaiwa (1987) tried to apply a simplified version of the "mapping instruction" (Resnick, 1982) between abacus operation and written calculation and found that it did not work well. Her subjects were twenty-six 3rd-graders who solved three-digit subtraction problems almost always correctly with an abacus, but often incorrectly with paper and pencil. In an instruction session, they were individually asked to solve each of those problems on which they had made errors in a pretest for written calculation, alternating the two computational procedures, i.e., solving with paper and pencil in steps 1, 3, and 5, and with an abacus in steps 2 and 4. In step 4, the subjects were required to subtract units first, then tens, and finally hundreds, i.e., in the reverseorder of the standard abacus operation, to make the mapping of the two procedures easier.

Two major patterns of responses were observed. Out of the 38 problems altogether, 24 were - + - + -, i.e., continuously making incorrect responses in written calculation whereas correct responses with an abacus; 11 were - + +, i.e., solving the problem correctly on an abacus enabled the subjects to answer correctly with paper and pencil and thus terminated the session. In the latter type, there seemed to be positive transfer, but results of a post test suggested that at least some subjects had just copied the answer obtained in step 2 which they believed to be correct. When they were given a new set of problems for written calculation, they tended to make the same types of errors as before. In sum, the subjects generally failed to repair their written calculation procedure by transferring knowledge about the abacus procedure. Another set of similar mapping instruction, attempted a few weeks later, did not work either.

According to protocols obtained by interviews, subjects who continued to rely on a "buggy" paper-and-pencil procedure while always giving the correct answer with an abacus believed that *perhaps* there was one correct answer for both procedures. They were not confident as to whether getting two different answers necessarily meant that an error was involved in at least one procedure.

Considering that Amaiwa's subjects had learned abacus skills fairly well, it may be surprising that they failed to repair their paper-and-pencil procedure by relying on the isomorphism between the procedures. This instruction may have failed because the students did not understand the meaning of each step of the "base" abacus operation and thus could not derive specific pieces of information to repair the "target" writing procedure. In abacus operation, as pointed out by Hatano (1988a), a set of specific "productions" (condition-action pairs, for example, "If addend 6 cannot be entered, subtract 4 from the target column and add 1 to one column left") replaces a general "production" (like "If an addend needs more beads than available, subtract the complementary number-to-10 of the addend from the target column and add 1 to one column left"). A few of these specific productions are then merged into a single production to get the final state directly (e.g., "If 7 is to be added to 6, leave 3 at the target column and add 1 to one column left"). Therefore, it becomes harder and harder to unpack the operation and find the meaning of each step. In other words, abacus operation as executed by experienced users is semantically opaque, that is, symbols and referents are not clearly connected during operation.

This semantic opacity was revealed more directly in interviews about the logic of specific steps with children learning abacus skills (Amaiwa & Hatano, 1983). Those who had had a year of practice at an abacus *juku* could explain the multi-digit subtraction procedure no better than their agemates who had just started the practice.

Effects of Abacus Learning on Written Calculation

Despite difficulties in mapping the two sets of knowledge, practice in abacus operation has considerable effects on written calculation, as revealed by Amaiwa & Hatano (in press). Third-graders who had been learning abacus operation at an abacus *juku* were not only much faster in basic computation (i.e., single-digit addition and finding complementary numbers-to-10) than their classmates who were not going to an abacus *juku*, but also much better in performance on paper-and-pencil multi-digit subtraction problems under a lenient time limit. Furthermore, the former group of children solved significantly more often than the latter open-sentence problems (e.g., "— + 8 is equal to 41," "— -7 = 27") and writing-an-expression for word problems (e.g., "There are 21 boys and 18 girls in Takashi's class. How many pupils are there?"). To eliminate unknown differences between the learners and non-learners statistically, we entered school grade in language as a covariate in the above comparisons. Since there were no differences in grade in science or social studies when the grade in language was entered as a covariate, it is reasonable to claim that this quasi-experimental design worked pretty well and that the above differences were due to transfer from abacus learning. Effects of abacus learning upon paper-and-pencil calculation were thus fairly far-reaching and substantial.

Why did such transfer occur? I infer that it was produced not through facilitated conceptual understanding, but through proficiency in shared subskills of simple computation. Proficiency in simple computation is a plausible candidate, because (a) a number of previous studies as well as the present one reveal that abacus learning greatly enhances it, and (b) eliminating its effect in statistical analyses makes other differences negligible.

I believe that component skills trained in abacus learning are used in written calculation. This is because these skills consist of sets of productions, and each production is "fired" more or less automatically—whenever its condition is satisfied, the corresponding action is executed. Since abacus-learning students are very good at these skills, they can concentrate on higher-order processes including monitoring of the steps of executive strategies and checking answers in a way suggested by Case (1982). It may also be possible for them to constrain their problem solving by quickly estimating the answer at sight. Another possibility is that their confidence in their ability to solve multi-digit subtraction makes them less "biased" toward addition than non-learners, when subtraction is in fact required. (Most of the errors the non-learners in this study made for open-sentence and writing-an-expression problems were using addition when subtraction was required; the reverse was rare.)

On the other hand, assessed comprehension of the place value principle or trade between columns, which is the conceptual basis of the borrow-and-decrement procedure in both abacus and written calculation, was not enhanced by abacus learning. When required to judge whether paired sets of numbers, expressed in terms of units, tens and hundreds, were equal or not (e.g., comparing [9 tens and 9 units] with [8 tens and 10 units]; [8 hundreds, 2 tens and 6 units] with [7 hundreds, 11 tens and 16 units]), abacus learners and non-learners performed equally poorly. Even when used as an additional covariate, the assessed comprehension did not change at all the observed differences in other tests.

Why did practice in procedural skills of abacus operation have no effect upon this conceptual understanding? I interpret this to be also due to the semantic opacity of abacus operation. In addition, practice on an abacus does not require or encourage conceptual understanding, because the same instrument is used throughout without changing constraints, and speed is emphasized (See Hatano & Inagaki, 1986). As a result, abacus operators apply the trade principle thousands of times without consciously recognizing it.

Abacus Operation as a Form of Non-School Mathematics

Abacus operation in Japan and street mathematics in Brazil (Carraher, et al., 1985) seem to have much in common: (a) both are used almost exclusively for commercial activities; (b) both can be acquired without systematic instruction; (c) both are outside of the "official knowledge" taught in school. In short, both are forms of non-school mathematics. However, the two are radically different in semantic transparency, i.e., how clear the meaning of each step of calculation is. Steps of street mathematics are clear in meaning because representations manipulated in it are information-rich, and ways of manipulation are analogous to actual activity dealing with goods or coins and notes. For example, in order to find the price for twelve lemons of Cr$ 5.00 each, a nine-year-old child who was an expert street mathematician counted up by 10 (10, 20, 30, 40, 50, 60) while separating out two lemons at a time (Carraher, et al., 1985). Quite to the contrary, representations of numbers on an abacus, though visibly concrete, are impoverished in meaning, and the way of manipulation is just mechanical.

Where these differences come from is an interesting question that merits close examination, but let me discuss it just briefly (See Hatano, 1988b, for more detailed analyses). I think the differences in semantic transparency can be primarily attributed to their functions in and the nature of the commercial activities they serve. Street mathematics is a means by which a vendor and a customer reach an agreement that calculation is done right. It is an interpersonal enterprise that requires semantic transparency—otherwise the customer may be suspicious. This semantic transparency also serves to make calculation accurate. It cannot be very quick, because it manipulates rich representations. However, the economy in which young Brazilian vendors live does not usually give priority to high efficiency in calculation. In contrast, abacus operation is basically a solitary activity, handling large numbers quickly and accurately. Operators are not interested in the semantic transparency of the calculation process, because they believe that their skills ensure the correctness of the answer. Even when abacus operation is used in interpersonal situations of buying/selling, both a vendor and customer are willing to accept the answer in most cases, as they trust the skills. In fact, a majority of Japanese customers seem to think that abacus operation is more dependable than calculation with a calculator. Experienced abacus operators must be able to handle impoverished representations, because the economy in which abacus operation developed required efficiency.

We can derive two instructional implications from the analyses above. First, we should not underestimate the significance of practice in basic computation for the development of mathematical cognition. Abacus learning is seen as having very limited instructional value, once we accept the premise that comprehension is much more important than proficiency. However, the present data indicate that enhanced proficiency in basic skills has wider effects than usually expected. It seems that "practice for proficiency in skills has its place" (Brownell, 1956, p. 129) in mathematics education.

Considering the semantic opacity of abacus operation, it is highly unlikely that its practice induces mathematical intuition (Resnick, 1986). However, abacus learners tend to do well in school mathematics, especially during elementary school years. This "success" may give them confidence in their mathematical ability; they tend to be free from "math anxiety."

Second, there can be a variety of non-school mathematics, each relating to school mathematics in its own way. Not every maths routine that emerges in non-school settings can serve as a basis for understanding how and why the corresponding "school maths" routine works. I very much like the basic idea of Carraher et al. (1985): mathematics learning in daily life produces effective and meaningful procedures which can complement potentially richer and more powerful mathematical tools acquired in school at the expense of meaning. However, daily life math can in fact make school math meaningful only when semantically transparent.

"Our lives are filled with procedures we carry out simply to get things done" (Hatano & Inagaki, 1986, p. 266). Both adults and children perform at least some everyday problem-solving procedures only because they "work," without bothering with the meaning of each step. Subtraction using an abacus, like pressing a key of a calculator for finding the square root of a given number, can be considered as one of such procedures. After repeating them hundreds of times, we can be quite skillful, i.e., we can become "routine experts." Despite this routine character, their component skills can be transferred to other procedures, since the actions of the specific condition-action pairs constituting a component skill are triggered automatically whenever the conditions are met (Anderson, 1982). Thus practice in one set of skills may facilitate the acquisition and performance of others sharing some component skills.

In short, the development of mathematical cognition can be conceptualized as a process of interaction of non-school and school math procedures. The nature of this interaction varies according to characteristics of the non-school procedures. To clarify the process, more research, particularly from a cross-cultural perspective, is needed.

Note

[1]This article is based on the paper presented at the symposium, "Mathematical concepts: learning and development in cross-cultural perspective," 9th Biennial Meeting of the International Society for the Study of Behavioral Development, July, 1987, Tokyo. I would like to thank Peter Bryant, Terezinha Carraher and Ed Hutchins for their valuable comments on its earlier version.

References

Amaiwa, S. (1987). Transfer of subtraction procedures from abacus to paper and pencil. *Japanese Journal of Educational Psychology,* 35, 41-48. (In Japanese with English summary)

Amaiwa, S., & Hatano, G. (1983, September). *Comprehension of subtraction procedures by intermediate abacus learners.* Paper presented at the 47th Annual Convention of the Japanese Psychological Association, Tokyo. (In Japanese)

Amaiwa, S. & Hatano, G. (in press). Effects of Abacus Learning on 3rd-Graders' Performance in Paper-and-Pencil Tests of Calculation. *Japanese Psychological Research.*

Anderson, J. R. (1982). Acquisition of cognitive skills. *Psychological Review,* 89, 369-406.

Brownell, W. A. (1956). Meaning and skill—Maintaining the balance. *Arithmetic Teacher,* 3, 129-136.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology,* 3, 21-29.

Carraher, T. N., Carraher, D. Q., & Schliemann, A. D. (1987). Written and oral mathematics. *Journal for Research in Mathematics Education*, 18, 83-97.

Case, R. (1982). General developmental influences on the acquisition of elementary concepts and algorithms in arithmetic. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*, (pp. 156-170). Hillsdale, NJ: Erlbaum.

Hatano, G. (1988a). *How are abacus beads different from coins and TILES?* Paper presented at the American Educational Research Association Meeting, New Orleans.

Hatano, G. (1988b). Becoming an expert in mental abacus operation: A case of routine expertise. *Advances in Japanese Cognitive Science* (Vol. 1), 141-160. Tokyo: Kodansha Scientific Publishers. (In Japanese with English summary)

Hatano, G., & Inagaki, K. (1986). Two courses of expertise. In H. Stevenson, H. Azuma, & K. Hakuta (Eds.), *Child development and education in Japan*, 262-27). New York: Freeman.

Resnick, L. B. (1982). Syntax and semantics in learning to subtract. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*, 136-155. Hillsdale, NJ: Erlbaum.

Resnick, L. B. (1986). The development of mathematical intuition. In M. Perlmutter (Ed.), *Minnesota Symposium on Child Psychology* Vol. 19, 159-194. Hillsdale, NJ: Erlbaum.

Selling Candy: A Study of Cognition in Context

Geoffrey B. Saxe
University of California, Los Angeles

This paper is concerned with the way children's participation in cultural practices can influence their developing mathematical understandings. My focus is on the practice of street vending, an activity common for unschooled children in developing countries. The vendors I will describe are 10- to 12-year-old boys who sell candy and are from poor urban areas in Brazil's Northeast.

The study is guided by a basic assumption: children construct mathematical understandings in their efforts to achieve mathematical goals, goals that often emerge as an interplay between their own prior understandings and the practices in which they participate (see Saxe, Guberman, & Gearhart, 1987 for an elaboration). This assumption leads to the view that analyses of children's understandings should be coordinated with and in part directed by socio-cultural analyses of the character of their practice-linked goals. In Table 1, I have sketched the structure of a two-part study with the candy sellers that follows from this view. On the left side of the table are basic questions I addressed, and on the right, the methods used.

0278-4351/89/1&2-19 $1.00 © LCHC

Table 1

QUESTION	METHODS OF STUDY
1. How are children's everyday mathematical goals influenced by participation in the candy selling practice?	Observations of sellers conducting their practice
2(a) What are the characteristics of sellers' mathematics?	(a) Interviews with sellers on mathematical tasks
2(b) Is sellers' mathematics influenced by participation in the candy selling practice?	(b) Contrasts between the mathematical understandings of sellers and matched non-sellers

My first concern represented in the table was to understand the form children's mathematical goals take in the candy selling practice. To answer this question, I conducted a series of ethnographic studies of the practice focusing on social processes that influenced the form of sellers' mathematical goals. The second question indicated was to discover the characteristics of candy sellers' mathematics. To address the second question, I conducted interviews with individual children using practice-related mathematical problems and contrasted the understandings of sellers with those of both urban and rural nonsellers. By contrasting sellers with nonsellers, I was able to determine whether practice participation affected the kinds of mathematical understandings children developed (see Saxe, in press, for a complete presentation of these data). I am going to summarize briefly the results of both types of investigations, focusing first on the ethnographic studies.

Ethnographic Analyses of Mathematical Goals that Emerge in the Candy Selling Practice

The sellers that were the target of study are entrepreneurs. Their practice has a 4-phase cyclical structure and in each phase they are likely to form particular kinds of mathematical goals. During a *purchase phase*, a seller buys a wholesale box of candy containing from 30 to 100 units from one of the many wholesale stores in a downtown urban center. During a *prepare to sell phase* a seller must translate the wholesale price for the multi-unit box into a retail price for just single units of candy. During a *sell phase*, a seller must sell his goods to customers. Finally, in a *prepare to purchase phase*, a seller must determine which of the many wholesale stores has the best price for the best box. While the basic cyclical structure provides a general context for mathematical goals to emerge, sellers' goals take form in and are thus influenced by a variety of social processes including macro-social processes like *inflation* and micro-social processes specific to the practice like *retail pricing conventions* and *social interactions*.

Inflation

Brazil has had a high inflation rate for many years—during the year of the study the inflation rate was about 25%. As a consequence of inflation, the mathematical goals that sellers construct in the practice involve inflated or very large numerical values, and the tokens of currency for these numerical values continue to shift as prices rise. For instance, the wholesale prices for boxes of candy ranged from about Cr$3600 to Cr$12000 when the study began and four months later they ranged from Cr$6500 to Cr$20000. The inflated monetary system has also meant that the government issues new denominations of currency frequently. Just before the study began, the government issued a 50000 bill—during the study, the government issued new coins in values of Cr$200, and Cr$500, and issued a new bill of Cr$100000, and, just after the study was completed, the government altered the currency system by eliminating three zeros from the cruzeiro, and calling the new unit the cruzado. This, because of the inflated currency, at all phases of the practice, sellers are dealing with very large numerical values as they generate arithmetical goals.

Social processes specific to the practice. Within the four phases of the practice, two types of social processes enter into the emergence of mathematical goals—sellers' mathematical conventions and sellers' typical social interactions.

Sellers' social conventions and their influence on sellers' mathematical goals. Over the history of the selling practice, sellers have developed social conventions that facilitate the conduct of their practice. Some of these conventions affect the nature of the mathematical goals that emerge in the practice. For instance, sellers use a retail price ratio convention in which they offer their candy in the sell phase to customers for multiple units for a single bill denomination, such as three chocolate bars for one thousand cruzeiros (Cr$1000). The convention reduces the complexity of arithmetical problems that would emerge in the sell phase if units were sold for odd numerical values—values that would give rise to time consuming and complex problems involving adding and subtracting bills. In turn, these very conventions give rise to new goals involving ratio comparison—many sellers, particularly the older ones, offer their candy for more than one ratio, such as 2 bars for Cr$500 and 5 for Cr$1000, and in determining these ratios, sellers compare the relative profit gained.

Sellers' social interactions and their influence on sellers' mathematical goals. At each phase of the practice, sellers have transactions with other people and, like the practice-linked conventions, these social interactions often give rise to or lead to an alteration in the mathematical goals of the practice. For instance, in the prepare to sell phase, sellers may negotiate with one another in price setting interactions to minimize local competition, interactions that may entail forming additional ratio comparison goals. In the sell phase, sellers may bargain with customers resulting in renegotiation of the pricing ratios, renegotiations in which sellers may need to form, again, new arithmetical and ratio comparison goals. In the purchase phase, wholesale clerks may help simplify sellers' mathematical goals by telling children box prices so that they do not need to read large numerical values as well as help with the translation of wholesale prices into retail prices.

Studies on Sellers' Mathematics

So far I have attempted to show that as a part of practice participation, sellers construct three principal goals involving the representation of large numerical values, arithmetic with large numerical values, and ratio comparisons. Now, I am going to address the question of whether sellers' engagement with the candy selling practice influences the character of their mathematical understandings. To address this question, children with little or no schooling from three groups were interviewed on mathematical problems related to the practice. The groups included 10- to 12-year-old candy sellers, 10- to 12-year-old nonsellers from the same urban environment, and 10- to 12-year-old nonsellers from a rural community about 100 miles away from the urban community. All children were presented with problems involving the representation of number, arithmetic, and ratio comparisons.

Expectations about children's performances were based on analyses of differences in the kinds of mathematical goals with which children are engaged in their everyday activities. Children from each of the three population groups engage in some commercial transactions in activities like running errands for their parents to grocery stores. This means that children from all groups are frequently faced with representing large values of currency. However, nonsellers do not often engage with problems involving arithmetic with multiple bill values, and problems involving ratio comparisons are even further removed from the activities of the nonsellers, especially the rural nonsellers. Because of these population group differences in the kinds of goals children structure in their everyday activities, I expected to find related differences in the character of children's mathematics.

Representations of Large Numerical Values

I used two types of tasks to assess children's ability to represent large numerical values in problem solving. In *Standard Orthography* tasks, children were required to read and compare multi-digit numerical values. As an alternative to the use of the standard orthography, I suspected that children might be using currency bills themselves as a basis for number representation. To test this hypothesis, I constructed *Alternative Representation* tasks that consisted of a *Standard Bills* condition in which children had to identify the numerical values of bills that varied in value from Cr$100 to Cr$10,000; a *Number Occluded* condition in which they had to identify the numerical values of identical bills with the printed numbers on the bills covered with tape so the child could not attempt to read them; and a *Numbers Only* condition in which children had to identify copies of the printed number on the bills without the bills' figurative characteristics. If children were identifying bills on the basis of their figurative characteristics, then children should perform better on both the standard bills and number occluded conditions than on the numbers only condition. This is just what I found. For the Standard Bills and Numbers Occluded conditions, there were no group differences—children from each group performed at or near ceiling; however, for each group, children made significantly fewer correct identifications on the Numbers Only condition. These results then indicate that children across population groups had developed an ability to use bills themselves as signifiers for large values and did not need to rely on their imperfect knowledge of the standard number orthography.

A counter interpretation of these findings is that perhaps children treat the values they use to identify bills as merely linguistic terms and do not order them as a numerical series. To evaluate children's knowledge of the numerical relations between currency values, children were presented with *Currency Comparison* tasks in which they were presented with pairs of bills and coins and asked to determine which was the greater value and then to determine how many of the lesser values was equivalent to the greater value (e.g., Cr$200 bill vs Cr$1000 bill). An analysis of children's responses to these currency comparison tasks revealed that children across population groups correctly identified the larger of the two valued currency units with great regularity. Children's answers to questions about the numerical relations between currency units revealed that all groups achieved a near ceiling performance on this task. Thus, not only do children identify bills without reference to the standard orthography, they also know both ordinal and cardinal relations between currency values.

Solutions to Multi-term Arithmetic Problems

The second area targeted for study was arithmetic with large currency values, a problem type more exclusively linked to the everyday activities of the candy sellers. To assess arithmetical competence with large bills, I presented children with a variety of tasks—tasks like adding a stack of 12 currency bills to Cr$8600 or subtracting Cr$3800 from Cr$5000. Typical strategies on this task involved "regrouping bills" into convenient values (Carraher, Carraher, & Schliemahn, 1985), strategies in which, for example, children would reorganize a series of bills to be added in an order in which they could add in multiples of 500 to 1000 cruzeiros. Unlike the *Alternative Representation System* tasks, children's performance varied across groups. Consistent with expectation, the sellers more frequently solved these problems than did the nonsellers.

Ratio Comparisons

The third type of problem targeted for study involved ratio comparisons, problem types that were not at all common in rural nonsellers' everyday activities, more common in the urban nonsellers' activities (who were sometimes the customers of the sellers), and, as we have seen, very common in the candy sellers' activities. To assess children's ability to compare ratios, an interviewer presented children with problems in which the child had to determine in which of two pricing ratios a child would make a larger profit (e.g., selling 1 candy for Cr$200 vs. selling 3 candies for Cr$500). Children were then required to tell which ratio would yield the larger profit and to explain their answers. Children who achieved accurate ratio comparisons typically justified their answers by constructing a common term, a construction that entailed transforming the numerator or denominator of one or both ratios so that it was comparable to the other. As expected, children's solutions varied markedly across population groups: Sellers' typically identified the appropriate pricing ratio and provided appropriate justifications for their answers whereas few nonsellers (especially the rural nonsellers) provided such responses.

Concluding Remark

The findings of this study add to our understanding of the processes by which children's participation in cultural practices influence their developing understandings. Mathematical problems are generated as children participate in cultural practices like candy selling. These problems are linked both to more general socio-economic processes like inflation, conventions like pricing ratios that arise in practices, and patterns of social interaction. As children participate in practices, their goals become interwoven with these social processes. In their efforts to achieve practice-related goals, children construct new understandings and solution strategies, new cognitive developments at once linked to their own constructive efforts and to social life.

References

Carraher, T. N., Carraher, D., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology*, 3(1), 21-29.

Saxe, G. B. (in press). The mathematics of child street vendor. *Child Development.*

Saxe, G. B. (forthcoming). *Cognition in context.* New Jersey: Erlbaum.

Saxe, G. B., Guberman, S. R., & Gearhart, M. (1987). Social processes in early number development. *Monographs of the Society for Research in Child Development.* No. 216. 52(2).

Note

This paper was presented as part of a symposium at the 1987 Meetings of the International Society for the Study of Behavioral Development in Tokyo, Japan. The paper is based on work supported by the National Science Foundation under Grant No. BNS 8509191 and by a small grant from the Spencer Foundation administered through the Graduate School of Education, UCLA. A more detailed analysis some of the findings described here is presented in Saxe (in press), and a more general description of the larger project from which this work is drawn is contained in Saxe (forthcoming).

A Content-Oriented Approach to Research on the Learning of Mathematics and Natural Language

Gerard Vergnaud
C.N.R.S. Paris

It is somewhat trivial to say that learning depends on the contents of knowledge to be learned. But many theories of learning have tried to get rid of the contents, with the aim of reaching the stage of general theories. This is the case with many structural theories as well as with the associationistic theories. Beyond the fact that general theories fail to help teachers in understanding the difficulties met by students for specific concepts and specific competences, it is theoretically disputable that knowledge develops along the same kind of process for biology and history, physics and mathematics, or even the geometry of the triangle, and the geometry of space.

I have two main arguments for this. First, empirical studies show, even in the limited domain of mathematics and physics, that the main difficulties met by students depend heavily on the contents of the situations to be mastered. Second, the search for general theories misses a very important epistemological point, namely that *concepts and competences are solutions to specific problems* that human beings have been faced with at one time or another. In other words, beyond the general consideration that knowledge always consists of concepts and competences, based upon properties that can be expressed in the same natural language or the same symbolic shape (graphs, tables, equations, etc.) we must not miss the essential point that every piece of knowledge refers to situations to be mastered or problems to be solved. Consequently, a developmental approach to knowledge and learning requires definitions that should enable us to deal with the situations for which a concept is meaningful. *We need a theory of reference that refers concepts to situations.* To apply this idea immediately, instead of giving definitions right now, I will start with examples.

0278-4351/89/1&2-2 $1.00 © LCHC

Additive and Multiplicative Structures

An important part of my research work concerns the development, the learning, and the teaching of the four operations: addition, subtraction, multiplication, division (Vergnaud, 1981; 1983a; 1983b). But trying to understand the reasons why the complexity of so-called elementary-arithmetic problems is so varied, even when the arithmetical operation to be used is the same, has suggested to me the idea that the first step was to classify the situations in which one has to add, subtract, multiply or divide. This led to the definition of two conceptual fields: additive structures and multiplicative structures.

By additive structures I mean the set of situations whose handling requires one addition or one subtraction, or a combination of such operations. By multiplicative structures I mean something similar for multiplication and division. Additive and multiplicative structures are not independent of each other, but for the time being it is better to consider them separately. These definitions seem straightforward. But the outcome is not. The classification of addition and subtraction problems, for instance, results in a wide variety of relationships whose analysis requires different concepts like the concepts of measure, of state and transformation, of order relationships, of combination measures, combination of transformations and combination of relationships, of natural number and directed number, of binary and unary operation, of directed state, abscissa and algebraic value, and others.

The reasoning (or relational calculus) that students have to make to find the right arithmetical operation in each situation is so different that children usually fail to recognize that it is the same kind of problem, even when the arithmetic operation is the same. For instance:

Example 1: Susan has 8 dollars. She spends 3 dollars on cakes. How much does she have now?

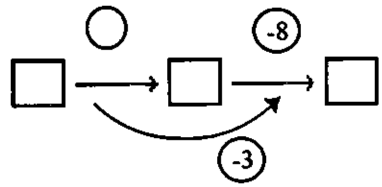
Find the final state.

Example 2: Peter has just played a game of marbles. He has won 3 marbles and he has now 8 marbles. How many marbles did he have before playing?

Find the initial state.

Inverting the direct transformation into a subtraction is not an easy operation of thinking for 7- to 8-year-olds. It even conflicts with their primitive conception of subtraction as a decrease.

Example 3: Robert has played two games of marbles. He lost 8 marbles in the second game, but he does not remember the first game. When he counts his marbles in the end, he finds 3 marbles less than what he had before playing the first game. What happened during the first game?



Find the first transformation.

There are two main difficulties in this case: you need reasoning without any information on the initial or final states; and you must subtract the whole loss from the second loss: the whole from the part, which is counter-intuitive.

I will leave additive structures for awhile because I would like to pay more attention to multiplicative structures.

Nearly all problems involving a multiplication, a division or a combination of such operations have to do with proportion: either a simple proportion between two variables, or multiple proportion of one variable to two or more independent variables.

Example 4: Eric wants to buy 4 miniature cars. They cost 3 dollars each. How much will he have to pay?

Can be analyzed in three different ways:

(a) binary operation $4 \times 3 =$
which is fair enough if you have pure numbers in mind; but if you think of 4 cars and 3 dollars, you can't explain why

multiplying cars by dollars gives dollars and not cars. However, if you keep thinking of the proportion between quantities of cars and quantities of dollars, conceptual differences between possibilities can be demonstrated through these two diagrams.

- binary operation $4 \times 3 = \square$

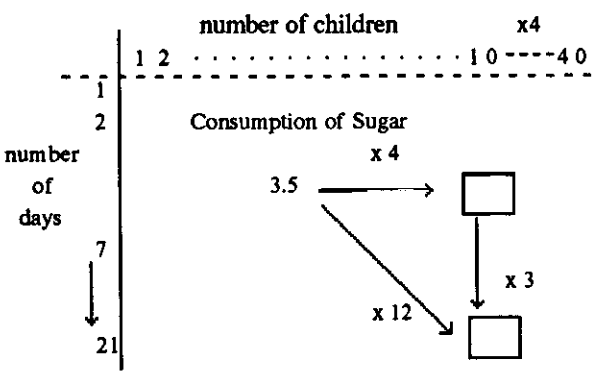
(b) x4 appears as a scalar operator that applies to 3 dollars and gives dollars.
x4 expresses the relationship between two quantities of the same kind (4 cars and 1 car). It can be expressed in natural language by "4 times more," which is purely scalar.

(c) x3 appears as a function operator that applies to 4 cars and gives dollars; x3 consists of a quotient of quantities "dollars per car." It cannot be expressed in natural language by "3 times more" and dimensional analysis can be used to show teachers (not children) the complete meaning.

$$\begin{tikzcd} 1 \arrow[r, "\times 3"] \arrow[d, "\times 4"'] & 3 \arrow[d, "\times 4"] \\ 4 \arrow[r, "\times 3"'] & \square \end{tikzcd}$$

Example 5: 40 children go to a holiday camp. They will stay 21 days. The average consumption of sugar for 10 children 3.5 kilograms per week. What quantity of sugar will they eat during their stay?

I have represented in a table the double proportion of the consumption of sugar to the time and to the number of children and one of the possible solutions of the problem.



—4 times more children and 3 times more days makes 12 times more

This solution lies on an essential property of the bilinear function: consumption of sugar as a function of the number of children and the number of days.

In the above example concerning prices of cars, the scalar procedure uses the isomorphic property of the linear function.

$$f(\lambda x) = \lambda f(x)$$
Price of 4 cars = 4 times the price of 1 car

and the function procedure uses the constant coefficient property of the linear function.

$$f(x) = a x$$
Price of 4 cars = dollars per car x 4 cars.

In contrast, the sugar consumption example involves a property of the bilinear function.

$$f(\lambda_1 x_1, \lambda_2 x_2) = \lambda_1 \times \lambda_2 f(x_1, x_2)$$

The consumption of 4 times 10 children during 3 times 7 days is 12 times (4 x 3) the consumption of 10 children during 7 days.

Of course students do not know explicitly these axioms and theorems of linearity but they do use them in solving problems. I call them "theorems in action," and can trace the use of theorems in action both in additive structures and multiplicative structures. There is a large variety of such theorems. Some of them are discovered or understood at an early stage in cognitive development (by 3 or 4 years of age for some local properties of addition and subtraction). Some are still difficult for most 16 or 17 year-olds and adults. If you can consider the different kinds of multiplication that I have just described, it is clear that the multiplication 4 x 3 = 12 cannot give account of the conceptual difficulty of the different problems presented.

Contents of Knowledge and Natural Language

I would like to raise now the problem of natural language. Language, especially in problem-solving, has two correlative functions: *communication* and *representation*. The communication function raises the question: who speaks to whom? The representation function: what is the speaker talking about? Other questions concern the nature of the task and the circumstances. None of these questions is independent of the others. Schneuwly and Bronckart (1983) have made the distinction between "theoretical discourses" and "discourses in situation," by arguing that some discourses are independent of the situations and do not contain the linguistic markers that refer to the *hic et nunc* situation: persons and objects, location, time, essentially. These markers such as pronouns and adverbs (he, this, here, before, now, after, etc.) are very frequently in "discourses in situation" and make some of them totally incomprehensible for persons who don't know the situation. This is the case when you listen to a tape, for instance. They also describe a third category of texts, narrative discourses, which describe the situation referred to.

When you observe students solving problems or asking questions about a situation, it is most likely that their verbal productions will be of the "discourse in situation" kind, although they may occasionally produce theoretical discourses (independent of the situation) and narrative discourses (referring to other situations). Language can be either a help in conceptualizing a situation or a help in learning and executing a sequence of actions, especially mental actions; it can also be both at the same time. This double function of language in problem-solving is not specific of natural language, it is also the sort of assistance you can get from a diagram, a picture, a table, an equation or a graph. It is different to put a problem into equations or put it into words, and to manipulate algebraic symbols or reason with words; but the function of representation concerns both algebra and natural language. The "4 times more" expression that I have mentioned before can be used by a student when he explains how to find the solution to another student or to the teacher. I have also stressed that it *represents* a scalar relationship and not a function. I would like to take other examples to show *how language can help controlling action and conceptualizing a new situation.*

The information given to Charlotte (2nd grade) is:

"Veronique has bought 24 postcards. She has written to her friends. She now has 11 postcards left."

No question is asked. Charlotte is supposed to formulate a question and then give the answer. She can use chips to solve the problem.

Charlotte (Ch.) and Interviewer (Int.)

Ch.: One could ask how many she used. (Ch. takes the tokens and says:) I'm going to take 24 - 1, 2, 3 ... 24 (counts quite loud) and that's 24. Then I'm going to save 11 now, I'll take 11, 5 and 6, that's ... that's 10 already, now I'm missing one, and one more ... then there are 11 there, I'll check because 1, 2, 3, 4 ... 10, 11 (counting quite loud). Then I have 11 there, I'll write it down shows her notebook). I have now many left? 2, 4, 6, 8, 10, 12, 14 (groups the tokens in twos and counts quite loud). There are14.

Int: Are you sure?

Ch.: 1, 2, 3, 4 ... 13 (counts again quite loud). Oh, 13.

Int.: And what can you say?

Ch.: These are the 13 postcards that she sent.

One can notice that Charlotte asks the relevant questions without any hesitation. Then she uses the tokens to calculate. The control by language appears several times, in different ways:

- she makes explicit what she is doing
- she checks
- she makes the first step of the procedure explicit
- she writes it
- and she counts again to find the result (her first answer having been wrong).

The specific parameters of "enunciation" are visible:

- pronouns: I, me (e.g., I'll take 11, I'm missing one)
- temporal and spatial deictics: now, there
- designation operators: (that's 10 already).

They are typical of "discourse in situation."

The same information is given to Pascale (2nd grade).

Pascale (Pasc.:) Starts with an addition before formulating any question.

Finds
$$\begin{array}{r} 24 \\ +\ 11 \\ \hline 35 \end{array}$$

Int.: What are these 35?
Pasc.: That ... 24 plus 11, you can ...
Int.: The 35, are these the ones she sent?

Pasc.: No.
Int.: Why?
Pasc.: Because 24, that was before and then 11 that's afterwards.
Int.: Then, what do you have to do?
Pasc.: Well, I think that I have to take these tokens? and that I can ... . (Pasc. takes the tokens.) I'll take 24 tokens. (He puts the tokens down counting up to 24).
Int.: And what are you going to do now?
Pasc.: I'll take the tokens up to 11. (He takes 11 tokens.)
Int.: And now, what do you do?
Pasc.: And now I'll write that there were ...
Int.: How many tokens did you take?
Pasc.: I took 24.
Int.: And what did you do?
Pasc.: I took ... I took the tokens ... my 11 tokens and there I have left (counts whispering 1, 2, ... 12) and there I have 12 left.
Int.: I think that there is a small mistake.
Pasc.: (Counts again the two piles) 13.
Int.: What can you say? What phrase can you make?
Pasc.: Perhaps that will be that she sent 13 and then she has 11 left.
Int.: Do you think that this is the way?
Pasc.: Yes.
Int.: Would you know how to write the operation?
Pasc.: I wouldn't know it.

Pascale never formulates the question but uses both language and tokens to elaborate a correct representation of the problem and the solution. Her last sentence shows that her final representation is quite good.

These protocols have been collected by Daniele Morange for her thesis. One of the important findings is that most children that can formulate the relevant question can also find the answer. The answer and the question seem to go together.

Conclusion

These considerations have led me to a comprehensive definition of what a concept consists of. It is a triplet of three sets:

concept = (S, I, R)

S: the set of situations that make the concept meaningful
I: the set of invariants (or theorems-in-action) that characterize the variety of competences of students. These invariants are properties of the concept.
R: the set of symbolic representations that can be used to represent these properties and the situations.

In other words, I would call S the referent, I the signified and the signifier.

The core of cognition lies in the operational invariants that students gradually discover or appropriate. I call them theorems-in-action because, after all, they are theorems. But there is a large variety of them, much larger than what we have intended up to now. The variety of situations is still larger and the landscape is made more complex by the fact that not only does a concept refer to a variety of situations, but each single situation cannot usually be analyzed with only one concept. For multiplicative structures for instance, we need the concepts of quantity, scalar, multiplication, division, linear and nonlinear function, dimensional analysis, fraction, ratio, etc. Because a concept refers to a large set of situations and because there are a variety of invariants and a variety of symbolic expressions for these invariants, we need to study rather extensive *conceptual fields*.

The concept of *conceptual field* aims at cutting out fair-sized objects for research on the development, the learning and the teaching of concepts and competences. It relies upon a content-oriented approach and not upon a logical, a linguistic or a structural approach. A reliable description of the development of knowledge in students' minds cannot avoid the contents of knowledge.

A conceptual field consists, first of all, in a set of situations. We must classify them and analyze the relationships that constitute the core of each class of problems. We must experiment with them at different levels, under different conditions, for different values of the variables, and then try to understand the variety of procedures and behaviors of students. We must also experiment with different ways of representing them, analyze what is well represented in a certain symbolic system, and what is not, and then understand the words, pictures and symbols used by students.

For instance, I have studied extensively the concept of number line as a representation of numbers and differences, of dates and durations. The results show that the understanding and the use of the number line is also, like language, a big problem to be solved, which has a lot to do with the analysis of additive structures and the analysis of space. It is not mainly a problem of syntax, it is a conceptual problem. The same is true for other symbolic systems like equations and diagrams, tables and graphs.

Conceptions of students depend on situations they have met. Primitive conceptions of addition and subtraction, multiplication and division, are shaped by the first situations mastered by students.

But we must also analyze how students modify these initial conceptions, by facing new situations, identifying similarities and differences, widening the scope of the first operations and eventually rejecting erroneous conceptions. During that process, which covers a very long period of time (over twelve years for additive and multiplicative structures), there are many metaphoric processes, and a few conceptual revolutions; also many misunderstandings. It is only a detailed description that can really enable teachers to interpret what students do and to find the explanations (also the questions and situations) that may help them.

The meaning of knowledge is conveyed by practical and theoretical problems to be solved, provided they are real problems for students. This is of course a functionalist's point of view. It has probably some limits. But for the time being, it would be profitable, for research on education, to try to answer the following question: *"to which problem or problems, does a new concept, a new property of a concept, a new procedure, a new representation or a new formulation bring a solution or eventually a better one?"*

This analysis cannot be done without a careful and profound analysis of the contents of knowledge.

References

Schneuwly, B., & Bronckart, J. P. (1983). *Pour une psychologie du langage. Archives de Psychologie, 51,* (196), 155-160.

Vergnaud, G. (1981). *L'enfant, la mathematique et la realite.* Bern: Peter Lang.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective,* pp. 39-59. Hillsdale, N. J.: Laurence Erlbaum.

Vergnaud, G. (1983a). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes,* pp. 127-174. New York: Academic Press.

Vergnaud, G. (1983b). Why is an epistemological perspective a necessity for research in mathematics education? *Proceedings of the Fifth Annual Meeting of PME-NA,* pp. 2-20.

The Social Constitution of the Mathematics Province—A Microethnographical Study in Classroom Interaction

Jorg Voigt
Institute of Mathematics Education (IDM)
University of Beilefeld, West Germany

The microethnographical study reported here examines how mathematical meaning is routinely constituted in the social interaction between teacher and students. The study of the social nature and of the covert mode of these processes contributes to a better understanding of the resistance of educational practice to every attempt of change.

According to several quantitative studies the usual mathematics instruction in Germany usually appears in the form of frontal teaching (Hopf, 1980; with regard to international studies Hoetker & Ahlbrand, 1969): the teacher asks a question to which he knows the valid answer, one student answers, the teacher evaluates this answer, and so forth. The third step, evaluation, has a decisive role in constituting official meaning. The teacher does not simply give a lecture, but draws the students into active participation in the process of knowledge communication; in this process, evaluation works in the selection of those statements from the students' contributions which shall (not) be considered as academic knowledge (Streeck & Sandwich, 1979).

In this kind of classroom discourse one can uncover concealed and stereotyped patterns and routines (Andelfinger & Voigt, 1984; Bauersfeld, 1980, 1982, 1987; Bauersfeld & Voigt, 1986; Voigt, 1984a, b; 1985, 1986, 1987). On the one hand, the patterns and routines facilitate the "smooth" functioning of the classroom discourse: on the other hand, they produce unintended effects on the students' learning.

These analyses of concealed routines are based upon concepts from theoretical traditions as symbolic interactionism (Blumer, 1969; Goffman, 1959), ethnomethodology (Garfinkel, 1967; Mehan, 1979) and phenomenology (Schutz & Luckman, 1973). Common to these concepts is a certain constructivistic perspective (Mehan, 1981). Bauersfeld, Krummheuer & Voigt (1986a) have modified

0278-4351/89/1&2-27 $1.00 © LCHC

the relevant concepts in order to deal with teaching and learning of particular subject.

We use detailed descriptions and interpretations of video recorded classes to reconstruct patterns of interaction and routines. These lessons represent regular mathematics instruction in several forms. The records have been transcribed (Voigt, 1983). the differentiated modes of interpretation (Voigt, 1984a; Bauersfeld, Krummheuer & Voigt, 1986b) adhere to the standards of qualitative research (Erickson, 1986).

The study of patterns and routines allows the investigation of old problems in mathematics education from a new perspective. For example, in problem-solving situations in the regular mathematics classroom the interactively constituted process of problem-solving differs from the ideal, individually produced process of problem-solving. Typical patterns of interaction are in use (Voigt, 1985). Neither teachers nor students are aware of these interaction patterns although they produce them routinely. There are also typical and hidden patterns in use when introducing concepts (Voigt, 1984a). In this paper, we will describe a pattern of interaction typical of concept-introduction in mathematics teaching in Germany. This pattern has been reconstructed frequently in situations in which teachers attempt to call upon students' experiences as a starting point in the introduction of a mathematical concept.

In order to illustrate this pattern and a few routines, a scene from a video-recorded mathematics lesson will be presented (section 1) and interpreted (section 2). Then a more differentiated description of the pattern based upon some theoretical concepts will be given (section 3). Finally, I will present some conclusions (section 4).

A scene taken from a mathematics class

Context of the scene. Fifth graders (aged 11 to 13) are having their first introduction to probability. The teacher wants to use the students' non-academic ideas as a starting point. The topic is a gambling-game. At first the teacher is aiming at the systematic study of observed frequencies over a number of runs, called the frequency approach. The teacher wants the students to play the game at a later point but he doesn't say so. (Later on he will aim at the approach of Laplace referring to the same game.)

At the beginning of the lesson the teacher presents a box containing eight chips of various colors. In the box are five red, one yellow, and two green chips. The teacher calls the box an "urn" and puts the conditions and rules of the game on the blackboard:

Game: Take one chip out of the urn with your eyes closed. Note its color. Then put it back into the urn.
Stake: 10 Pfennig (Pfennig = cent)
Prize/Return: 20 Pfennig, if one takes the yellow or a green chip.

While the teacher is explaining the game, the students joke as if they intend to really play the game, placing bets and getting prizes. They display their cash. The transcript below begins at this point; the scene lasted three and a half minutes.

Transcript. This translation of a German transcript is a compromise between several aims: preserving the meanings, the structures of the spoken language, and using colloquial phrases and idioms (original transcript in Voigt, 1983; for the rules of transcription, see appendix).

75 T: now, that's the game (*walks toward the students, joining his hands*
76 what question could one ask about this game'
77 S: hello
78 S: where will the money come from (*Ss laugh*)
79 NS: let's grab.
80 T: leave the money keep the money back.
81 NS: what a pity
82 T: we're going to do it differently with the money
83 I don't want to pull your pocket-money out of your pants
84 (*walking around until line 114)*
85 Ss: oh, oh
86 T: but what question, what question could you ask about this, about
87 this game. (*points to the blackboard)*
88 S: here
89 T: Katrin.
90 Ka: how many possibilities does one have then generally there .. so
91 many chips can one (..)
92 T: could you raise your voice. I think that Ulf,
93 he didn't understand you.
94 Ka: how many possibilities does one have there so to win' .. of
95 seven chips one can, of one draws three of them can get
96 twenty Pfennig .. and the other four (..)
97 S: (*loudly)* five

98 S: three
99 S: three to five
100 Ma: the odds are seven to three
101 T: yeah
102 Ns: seven to three
103 T: so you're already thinking about the number of the chips and
104 S: eight
105 T: going to, going to ask then the question, yes how many
106 possibilities does one have actually, or how many does one have
107 S: seventy to thirty
108 T: generally of these here to win something. *(moves the left hand)*
109 yes, just easy..still a more fundamental question,
110 Stefanie.
111 St: yes how large is the probability to win there.
112 T: yes
113 St: twenty Pfennig
114 T: or, or who does win generally *(stand*
115 *still)*
116 Ns: probably
117 T: playing this game.
118 Ns: (..) pay for the stake.
119 T: who does win generally'
120 Nss: *(trouble)* there he, at all generally get (...)
121 T: therefore the question, does the player win or does the
122 gambling-bank win. which has e set up this game' *(writes the*
123 *question on the blackboard.)*
124 Nss: the bank, game, generally the bank wins
125 T: now, how can one check whether the player or the
126 bank wins' *(5 sec)* Andreas.
127 Ad: generally the bank wins. that's the same as with the
128 lottery tickets draw one of them you';re buying ten pieces,
129 really. about five of them .. generally .. are blanks and for
130 the rest then you get small prizes.
131 *(T rests his chin in his hand)*
132 T: yeah that's right. but let's assume we're betting here instead
133 of twenty Pfenning ten Mark *(points to the blackboard)* *(Mark =dollar)*
134 Ss: *(loud)* oh, oh, oh
135 S: *(T is listening with thoughtful expression)* that is a heavy
136 loss namely if, not so many people come there only a few and
137 each of them gets then always ten Mark and so on - (..)
138 T: therefore one can state, that then with certainty
139 the bank looses and the player wins..here, the prize
140 isn't that high, but it's only twenty Pfennig.
141 only in quotation marks, it is also something,
142 twenty Pfennig,
143 Ss: umm, umm.
144 T: and .. the question is still open.
145 does the player' win how could one check it out
146 if the player really wins or if possibly as Andreas
147 assumes, the bank, the bank always wins.
148 S: always not
149 T: Lars.
150 La: I think that the bank is more likely to win
151 Ns: more likely
152 T: this isn't working .. okay you can make guesses
153 *(rejecting wave of his hand)* will the bank win
154 *(walks around, Ss laugh)* who is for the bank or who does think
155 that the bank will win this game' *(several Ss raise their*
156 *hands)* .. who does think that the player, has a chance at this game, and wins'
158 Ss: *(many-voiced and loud)* a chance *(some Ss raise their hands)*
159 S: *(loud)* a chance he has got.
160 S: *(laughing)* but only a small
161 T: yes'
162 S: here
163: T: who does think, well, I'll say it in another way, who does
164 think that the player will win' .. *(fewer Ss raise their hands)*
165 there are some too. well..
166 S: Sir
167 T: okay we can, okay we can discuss it *((ausdiskuti eren))*
168 but we can also'
169 Ss: try *((probieren))*
170 Ma: try it. *((ausprobieren))*
171 T: yes. what can we do (..)
172 S: yes of ten times
173 T: yes easily try it
174 S: yes of ten times how many times the player will win
175 Ma: but generally will be red

176 NS: I don't understand
177 T: yes, so it's about to play the game *(Ss laugh)* and to try it.
178 *(lift the urn up)* we will do it as follows - .. there come
179 now five *(finger at his mouth, Ss laugh loudly)* there come
180 five .. students to me, a fairy of fortune we need. *(Ss*
181 *raise their hands)* a fairy of fortune
182 NS: Jens the fairy

Interpretation of a Sequence

The following interpretation is focused on a short sequence of the scene. The extensive scene gives the necessary context for the interpretation of the actions of the sequence: At the beginning in line 76 the teacher opens with an ambiguous question: "What question could one ask about his game." Obviously the teacher wants to hear that the students wonder whether the player or the gambling-bank will win, like in everyday life. But at first the students are not guided by everyday ideas. Rather their answers are guided by an assumed relevance to the mathematics lesson, as for example line 100: "the odds are seven to three" or line 101: "how large is the *probability* to win there." These answers go farther than the teacher wanted. Finally the teacher himself gives the answer he expected: "who does *win*" (line 114).

In line 125 the teacher asks a second key question: "how can one check whether the player or the bank wins." The teacher wants the game to be done repeatedly according to the frequency approach, i.e., after playing the game several times, the relative proportion of the summarized returns to the summarized stakes would be taken as the probability of winning without respect to the distribution of chips in the urn.

This turn of the sequence and the next ones will be interpreted more exactly:

- In the first turn (line 114-126) the teacher attempts to activate the students' non-academic ideas as a starting point. He asks an open question hoping to elicit the students' everyday ideas.

- In the second turn (127-131) a student refers to his own subjective experiences from everyday life. The student Andreas refers to the context of a lottery. Andreas mentions a worldly wisdom: the bank profits always. Andreas values the outcomes of the game as "blanks" and "small prizes." Both are negligible quantities for him. He doesn't aim at stating the proportions of gains but evaluates the outcomes under the perspective of the gambler. Nevertheless the teacher doesn't want this idea.

- In the third turn (132-133) the teacher rejects the students' everyday ideas. He modifies the topic, by temporarily modifying the level of the possible winning (not 20 Pfennig but 10 Mark). He modifies the topic in such a way that on one hand the everyday context is maintained at the surface and on the other hand the everyday context has to be transgressed in order to solve the problem. Under the teacher's perspective the structure of the game is not changed while Andreas' arguments become invalid.

These three turns form a pattern of interaction which shall be called *"pattern of the staged everyday reference"* or in short "staging pattern." The teacher starts a topic using an embodiment which elicits the students' subjective everyday experience. The everyday reference is staged in the sense that it is elicited and is immediately rejected. Prospects and orientations of the students' everyday experiences (here of a lottery) are not taken up by the teacher. The teacher cuts these experiences from the official constitution of the theme.

This staging pattern has been reconstructed in the lessons of different teachers frequently. It appears in situations in which the teachers attempt to activate the students' experiences as a starting-point, especially by means of embodiments (Voigt, 1984a).

Theoretical Background

Now I should like to take a closer look at the staging pattern by means of some theoretical concepts. These concepts were developed to understand discourse processes. These processes have a dynamic force of their own often departing from the teacher's intention.

How the dynamics of interaction form a pattern can be described by means of the concept of *"implicit obligations."* For example, in everyday greetings: the person who greets expects to be greeted in return, and the person greeted fulfills the implicit obligation to return the greeting. Such implicit obligations can be reconstructed in classroom interaction, too. Related to the staging pattern in our scene I reconstruct the following obligations:

- Because of the teacher's ambiguous, open question the students are under the obligation to express everyday ideas. For example it is improper for students to treat the teacher's questions as unclear or ask the teacher about his expectation with respect to the answer. Showing non-understanding in the classroom would be self-discriminating for the student. In our scene the student Andreas meets the obligation, but his answer doesn't correspond to the teacher's hidden expectation of the answer.

- The teacher feels himself under the obligation to accept Andreas' answer, on one hand, and to carry out the game, on the other hand, in order to attain the frequency approach. The teacher can neither say that Andreas' answer is wrong nor does he want to enter into Andreas' arguments. So he modifies a condition of the game spontaneously, a marginal condition as he thinks.

- Now the students feel themselves under the obligation to accept the teacher's modification. The students express their surprise in line 134, however in line 135 one student takes up the modification.

By such implicit obligations a pattern of interaction develops. The obligations can be considered partly with regard to the institution of school. For example, in other cases when the students began to modify the topic, the teacher used his authority to reject the students' modifications.

Also the students play their part. They know that the teacher will evaluate their answers. This expectation prevents them from insisting on their own conception although their conception is valid under other circumstances—like the worldly wisdom displayed in our example. Thus, teacher and students stage the illusion of a liberal instruction based on arguments.

The smooth functioning of the classroom discourse succeeds due to the staging pattern in spite of the differences in perspectives of teacher and students. As a prerequisite for mutual understanding a process of negotiation of meaning takes place (Cobb, 1988; Krummheuer, 1982, 1983a, b). The provisional willingness to cooperate can be described by Erving Goffman's concept of *"working consensus"* (1959) and Gotz Krummheuer's concept of "working interim" (1983a, see also "contract didactique," Brousseau, 1984). The working consensus is an implicit agreement. For example, our teacher evaluates Andreas' answer as virtually correct though it differs from his own objectives. In line 132 the teacher says: "yeah that's right, but ... ." The students treat their own offers as tentative, that is, they do not need to take full responsibility. Their tentative verbal actions are based on their trust that their answers and the teacher's questions will be subsequently clarified. By this working consensus the teacher quickly realizes his objectives while seemingly taking the everyday experiences as a starting-point.

As I spoke of obligations, of institutional pressure, and of the working consensus I didn't think of mathematics instruction as a rigidly stabilized or pre-stabilized enterprise. No teacher is safe from the students' creativity. But recurrent patterns of interaction can be reconstructed. These regularities can be explained by *"routines"* of the teacher and students. So the teacher and the students do not need to feel the obligations as compulsions or to feel the working consensus as a contract. Here I refer to Alfred Schutz and Luckmann's (1973) concept of routine and its use in ethnomethodological studies. A pattern of interaction is interactively produced turn-by-turn by means of everyday school routines.

The routines have interactive functions: for example, the students' routine of verbal reduction, that is the restriction of utterances to numbers and catchwords, enables the teacher to identify in the students' utterances the meaning which he expects, and vice versa. The routines of the teacher and of the students reduce the complexity of classroom discourse. The routines relieve the acting person and make the actions mutually reliable for the participants. In this sense routines are necessary (Bromme & Brophy, 1986).

However, one can investigate the (potential) disadvantages of certain routines (Voigt, 1986): In the described staging pattern the teacher uses the rhetorical trick of modifying conditions of a topic in order to reject students' arguments.

By means of this routine our teacher copes with one problem. On the one hand, he wants to use the students' everyday experiences as a starting-point, on the other hand, he aims at a certain introduction to probability. The teacher does not need to be aware of the tension between the students' subjective ideas and the teacher's own demands, especially in case he regards mathematics learning as a direct abstraction of everyday ideas. I reconstruct the teacher's actions in such a way that he wants to hear certain everyday ideas which he will take as a starting-point. In this sense the everyday reference is staged.

However, the students are not confronted with the mathematical model directly. They come to know that they are expected to express their ideas and they come to know that the teacher doesn't take their ideas seriously. Classroom interaction of the type of the staging patterns runs the risk that the student copes with this conflict by understanding mathematics as a strange, non-evident, irrational subject matter. This problem returns several times within the next lessons of this class. The students argue in similar situations like Andreas, and the teacher rejects their arguments until the game obtains a very odd character. While the teacher thinks that he uses the students' experiences as a starting-point, the students distinguish between everyday gambling games and the specific gambling games in mathematics lessons. The working consensus indurates.

Thus one can identify a gap in our scene, a gap between the teacher's intentions and his routines. He aims at a constructive connection between the relevant everyday experiences and the mathematical model. His routines lead him to a direct approach to mathematics and a cutting off of everyday experiences.

At the end of the scene the teacher achieves his objectives by using routines once again. In line 154 one can reconstruct the routine of the "tactical poll" and in line 167 one can reconstruct the routine of the "suggestive hint." The teacher makes the students say that the game should be tried—thus, achieving his objectives step by step. In doing so he might still believe that the students express and follow their own ideas.

Conclusions

Surely it is possible to criticize several teacher's actions in our scene and to construct better ones. The teachers themselves can do it when confronted with the transcript of their own classes. For example, one may say: The teacher's embodiment doesn't fit the frequency approach. Or: The teacher has to employ meta-communication in order to promote an explicit translation from the embodiment to the mathematical model. When contrasting the value of "20 Pfennig" as a small prize in everyday experience with the fact that the return is nevertheless twice the stake in the mathematical model the teacher himself gives the hint for this translation (line 139). But I am not out to criticize.

When replaying the video tapes and reading the transcripts together, the observed teachers again and again expressed their surprise about what was going on in the microprocesses. They did not realize the routines and the unintended effects. The routines are not only a problem of experienced teachers; they have been reconstructed also in the very first mathematics lessons given by beginners. Moreover, the staging pattern and actions like suggestive hints etc., have been reconstructed in Plato's dialogue "The Meno" between Socrates and the slave, in the oldest "document" of mathematical instruction (Struve & Voigt, 1988). It seems that the patterns of interaction and the routines are elements of a classroom culture which is reproduced by concealed socialization. For instance, the teacher's routines can be formed early by internalization of the student's role. This may be one of the reasons for innovations in teaching to work only in long-term basis.

In the context of teacher training it can be helpful to develop the teacher's awareness of the functioning of the microprocesses taking place in the mathematics classroom. The analyses of patterns and routines may contribute to the teacher's reflection. The reflection on what has been taken for granted can serve as a preparation for new approaches for teachers.

Appendix

Rules of transcription:

T	Teacher
S	student
Ss	students
NA	A in private communication
Ka	Katrin
Ma	Marc
St	Stefanie
Ad	Andreas
La	Lars
A: but then	}A and B speak partly at the same time
B: why is	}
A: but then	}B interrupts A
B: why is }	
,	very short pause (max. 1 sec.)
..	short pause (max. 2 sec.)
...	medium pause (max. 3 sec.)
(4 sec)	long pause
.	lowering the voice
'	raising the voice
-	maintaining the pitch
exact	emphasizing
exact	drawling
(whispering)	
(walking around)	manner of speaking, etc.
(...)	inarticulate utterance
(two?)	inarticulate, but probable utterance

References

Andelfinger, B., Voigt, J. (1984). Der Alltag des Mathematikunterrichts und die Ausbildung von Referendaren. Occasional paper 51. Bielefeld: IDM

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a methematics classroom. *Educational Studies in Mathematics*, 11, 23-41.

Bauersfeld, J. (1982). Analysen zur Kommunikation im Mathematikunterricht. In *H. Bauersfeld et al., Analysen zum Unterrichtshandeln.* Koln: Aulis.

Bauersfeld, H. (1987). Interaction, construction, and knowledge—alternative perspectives for mathematics education. In T. Cooney, & D. Grouws (Eds.), *Effective mathematics teaching.* NCTM, Reston, Virginia.

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1986a). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner (Ed.), *Proceedings of TME.* Bielfeld.

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1986b). Interaktionsanalyse von Mathematikunterricht—Methodologische Annahmen und methodisches Verfahren. Unpublished Paper. Beilefeld: IDM.

Bauersfeld, H., & Voigt, J. (1986). Den Schuler abholen, wo er steht! Inszenierung oder Verwirklichung eines didaktischen Prinzips? In *Jahresheft IV: "Lernen—Ereignis und Routine,"* pp. 18-20. Velber: Friedrich Verlag.

Blumer, H. (1969). *Symbolic interactionism: Perspective and method.* Englewood Cliffs: Prentice-Hall.

Bromme, R., & Brophy, J. (1986). Teachers' cognitive activities. In B. Christiansen, G. Howson, & M. Otte (Eds.), *Perspectives on mathematics education*, pp. 99-140. Dordrecht: Reidel.

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H. G. Steiner, et al., *Theory of mathematics education.* Occasional Paper 54, pp. 110-119. Bielefeld: IDM.

Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist*, 23(2), 87-103.

Erickson, F. (1986). Qualitative methods in research of teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching*, pp. 119-116. Third edition. New York: Macmillan.

Garfinkel, H. (1967). *Studies in ethnomethodology.* New Jersey: Prentice-Hall.

Goffman, E. (1959). *The presentation of self in everyday life.* New York: Doubleday.

Hoetker, J., & Ahlbrand, W. P. (1969). The persistence of the recitation. *American Educational Research Journal*, 6, 145-167.

Hopf, D. (1980). *Mathematikunterricht. Eine empirische Untersuchung zur Didaktik und Unterrichtsmethode in der 7. Klasse des Gymnasiums.* Stuttgart: Klett-Cotta.

Krummheuer, G. (1982). Rahmenanalyse zum Unterricht einer achten Klasse uber "termumformungen." In H. Bauersfeld et al., *Analysen zum Unterrichtshandeln* (pp. 41-103). Koln: Aulis.

Krummheuer, G. (1983a). Das Arbeitsinterim im Mathematikunterricht. In H. Bauersfeld, et al., *Lernen und Lehren von Mathematik* (pp. 57-106)

Krummheuer, G. (1983b). Algebraische Termumformungen in der Sekundarstufe I—Abschlubbericht eines Forschungsprojekts. Materialien und Studien Band 31. Bielefeld: IDM.

Mehan, H. (1979). *Learning lessons.* Cambridge, MA: Harvard University Press.

Mehan, H. (1981). Social constructivism in psychology and sociology. *The Quarterly Newsletter of the Laboratory of Comparative Human Cognition*, 3(4), 71-74.

Schutz, A., & Luckmann, T. (1973). *The structures of the life-worlds.* Evanston, IL: Northwestern University.

Streeck, J., & Sandwich, L. (1979). Good for you.—Zur pragmatischen und konversationellen Analyse von Bewertungen im institutionellen Diskurs der Schule. In J. Dittmann (Ed.), *Arbeiten zur Konversationsanalyse* (pp. 235-257). Tubingen: Niemeyer.

Struve, R., & Voigt, J. (1988). Die Unterrichtsszene im Memon-Dialog—Analyse und Kritik auf dem Hintergrund von Interaktionsanalysen des heutigen Mathematikunterrichts. *Journal fur Mathematikdidaktik*

Voigt, J. (Ed.). (1983). Mathematikunterricht im 5. bix 11. Schuljahr—Transkripte zum Projekt "Routinen im Mathematikunterricht." Materialinen und Studien Band 33. Bielefeld: IDM.

Voigt, J. (1984a). *Interkationsmuster und Routinen im Mathematikunterricht—Theoretische Grundlagen und mikroethnographische Falluntersuchungen.* Weinheim: Beltz.

Voigt, J. (1984b). Der kurztaktige, fragend-entwickelnde Mathematikunterricht—Szenen und Analysen. *Mathematica Didactica*, 3/4. 161-186.

Voigt, J. (1985). Pattern und routines in classroom interaction. *Recershes en Didactique des Mathematiques*, 6, 69-118.

Voigt, J. (1986). Sozial-interaktiv Bedingungen der Entwicklung mathematischer Fahigkeiten im gegenwartigen Mathematikunterricht. In H. G. Steiner (Ed.), *Grundfragen der Entwicklung mathematischer Fahigkeiten.* Koln: Aulis.

Voigt, J. (1987). An interactional approach to learning and teaching mathematics. Paper presented at the Research Presession of the Annual Meeting of the National Council of Teachers of Mathematics. Anaheim, CA.

Unevenness in Mathematical and Cognitive Development:A Discussion of the Five Papers

P.E. Bryant
Oxford University

While reading the five papers in this fine collection I was struck by the possible connections to be made between our current knowledge about children's informal mathematics and our current dilemmas with work on cognitive development. The thought that occurred to me was not so much that theories about children's cognitive growth can be used to help us understand the development of their mathematical skills: that, after all, is a connection which has been tried many times already. My idea took the reverse direction. It occurred to me instead that recent research on children's mathematics might well provide the answer to several difficult questions about cognitive development. The research is so interesting and its results so convincing that it may well have implications far beyond the issue of how children learn arithmetic, algebra and geometry.

To make this point I shall start with an extremely brief description of the current state of work on cognitive development. That work has been remarkably successful on the whole, but it has also provoked some stubborn problems. The modern history of the subject can traced back to demonstrations by several people, including Binet and Piaget (1952) that there are striking changes in children's intellectual skills as they grew older, and that the most surprising thing about these changes was the number of achievements which adults take for granted but which seem to be out of the range of quite normal children. To list all these apparently missing skills would take a long time and it would anyway be an unnecessary task, for they are well known. Piaget in particular based much of his work and his theoretical ideas around them.

His conservation task, the results of which led him to claim that children younger than seven years or so do not on the whole understand the principle of invariance, and his transitive inference problems, in which children apparently fail to conclude that A>C from the information that A>B and B>C, are probably the best known examples. In both cases young children apparently lack an understanding which is a basic and unquestioned part of an adult's intellectual repertoire.

If such claims are true then it only remains to work out how children eventually acquire the basic skills which they are said to lack while they grow older. In the relatively brief history of developmental psychology, the main candidates offered for this causal role have been Vygotsky's idea about the role of language and the zone of proximal development on the one hand (Cole, 1985; Vygotsky, 1962, 1978), and Piaget's equilibration theory on the other. A description of those two formidable causal hypotheses is beyond the scope of this brief commentary. The only point that I want to make about them here is that the main purpose of both is to explain how children acquire skills *de novo*—skills, that is, which they completely lack at first but which come to them as they grow older.

This concern with the acquisition of entirely new intellectual skills dominated developmental psychology for a long time, but there is now a new interest which is at least as widespread as the first. This arises from the growing evidence that the traditional picture of skills at first not being there, so to speak, and then arriving later on in childhood can be very misleading. It is now quite clear that young children can manage many of the achievements which were thought to be impossible for them, but that they can only manage them at some times but not at others and in some circumstances but not in others.

Conservation tasks and transitive inference problems happen to be good examples. Children who fail the traditional conservation task (Piaget, 1954) often succeed when given other closely similar problems which also seem to test their understanding of the invariance principle.

In my view the best illustration of this unevenness is the demonstration that many children who would normally fail the usual test which involves two questions (one before, the other after, the transformation) succeed when they are only asked one question (Rose and Blank, 1974). If they are shown two identical rows of objects but are asked nothing about them and then after one of the rows has been spread out, they are asked for the first time to compare it with the other one, they are often able to judge

0278-4351/89/1&2-2 $1.00 © LCHC

that the rows have equal numbers: they do so much more than if they were asked to make the comparison twice, once before and once after the transformation.

One can say much the same about transitive inferences. Measurement for example is based on transitive inferences, and it was thought for a long time that young children were incapable of using an intervening measure to compare two quantities: they tended instead to resort to direct comparisons between the two quantities in question without using a measure (Piaget, Inhelder and Szeminska, 1960). Now however it is known that young children do often use measures if a direct comparison between the quantities in question is impossible (Bryant and Kopytynska, 1976).

The importance of these two examples is that children seem to manage to use the principle of invariance or the principle of measurement in some contexts but not in others. Such data are obviously important, but they raise difficult questions. One has to try to explain why children, who apparently have a certain ability, decide to apply it to one situation but not to the other. Good explanations of this sort of unevenness in a particular task have been hard enough to find. General explanations of the same phenomenon over several different types of task pose an even more serious problem. Attempts at such hypotheses tend to be too general to produce specific and testable predictions. Hypotheses about 'access' seem like this to me. Children are said to have access to a particular strategy when they use it, and not when they don't. The idea itself does not tell one why one context is more suitable than the other.

What is the relevance of these questions to mathematical development in general and specifically to the five papers in this issue? The most obvious starting point for the connection that I am making is the fact that work on children's mathematical skills also reveals striking unevenness in the way children deploy their skills. One absolutely firm conclusion to be drawn from recent research on the way in which children learn about mathematical operations is that there is a great deal more to it than simply learning what these operations are. Children have to learn how to add, multiply and so on, but they also have to learn when it is appropriate to do so. There is now a great deal of evidence that this second kind of learning often causes children a lot of difficulty.

The evidence for this is the unevenness which characterizes their decisions about when to apply particular mathematical operations. Children who add and subtract perfectly well when answering some problems nevertheless frequently fail to resort to these same operations when given other equally appropriate problems (Bryant, 1985; Fuson, 1988).

To make this point is simply to say that things are much the same in the field of mathematical skills as they are in other branches of cognitive development. However, it seems to me that the papers in this issue not only reflect the familiar distinction between having a skill and knowing when to use it; they also give us a clearer idea about the nature of this distinction. I want to argue that they tell us more about two vital questions—which are, (1) why there is a gap between having a skill and being able to use if appropriately, and (2) how the gap is eventually narrowed and even sometimes closed.

Vergnaud's paper starts with a statement, and soon follows with an example, about the inherent unevenness that characterizes the way children apply mathematical strategies that they have learned and the possible reasons for it. His opening arguments—that the biggest difficulties in mathematics are caused by the contents of the problems and that people learn mathematical concepts in order to solve specific problems—add up to a reminder that the content of a problem and its context are as likely to determine whether it is solved by a child as the actual mathematical skills that he or she possesses. His simple demonstration that children who can perfectly well use subtraction to solve a problem about a final state (Susan has \$8 and spends \$3 on cakes. How much money has she left?) nevertheless fail to subtract when given an equivalent problem about a beginning state (Peter has won 3 marbles and now, has 8. How many did he start with?) is arresting. What is going on here? It seems likely that in the second, evidently much more difficult, problem the verbal context is the barrier (Carraher and Bryant, 1987). To be told that a boy has 'won' some marbles is to be put into a context that is naturally associated with addition, since winning something means adding it to one's possessions, just as spending money (the first example) typically means subtracting it from one's wealth. So the semantic context probably encourages subtraction in the first example and discourages it in the second.

There is a deeper point to be made about this important demonstration. If my argument about the semantic context is correct, the children who find the second problem much harder than the first are strongly affected by the nature of the actions involved. If spending means subtraction and winning means addition to them, then they probably think most clearly about these operations in terms of particular actions. Thus it might be that the unevenness that I have been discussing is, at least partly, caused by the kind of model that children adopt. If the model is in terms of actions, mention of some actions will lead to the appropriate strategy and of others to the inappropriate one.

Of course this explanation does not solve the problem of how to help children to apply their mathematical knowledge appropriately, and the problem is a grave problem, as the papers by Carraher and Meira and by Voigt show in different ways. Carraher and Meira deal with a controversial topic—the use of LOGO as a way of teaching children about mathematics and particularly about geometry. One of the main claims for LOGO is that it makes it possible for children to think about geometrical properties in general and about angles in particular in terms of their (the children's) own actions. The children have to trace lines by making a 'turtle' move around a screen, and the idea is that they can relate the turtle's movements to their own and thus form a better understanding of the geometry (angles, distances) involved in the instructions that they have to give to the turtle. Given what I have just said about the importance of actions in children's models of mathematics, the claim for LOGO is a particularly exciting one. Carraher and Meira's work, however, suggests that the claim is questionable. They point out that it simply cannot be taken for granted that children will make or even understand the analogy between the turtle's movements and their own. Moreover Carraher and Meira's data also show that many adolescents have very great difficulty in understanding the turtle's actions and in expressing these in terms either of the turtle's movements in space or of their own. Here is a definite failure to make a connection which several educators assume will be transparently easy for young children, but we can wonder whether it is to be put down to the children's inability to make connections in general or whether this particular connection is an especially hard one to make. The turtle's movements are expressed in terms of movements in particular directions and turns of particular angles (e.g. left 45 degrees, forward 100) from the point where the turtle happens to be when the instruction is issued, but it seems unlikely to me that children ever think of their own movements in space in terms remotely like this. Children moving in space will, presumably, be thinking explicitly in terms of their absolute destination, and it seems likely that the turns that they make and the distance that they travel are the product of implicit decisions of which they are wholly unaware.

So if children are to make a connection between everyday knowledge and classroom mathematics one should at least be sure at first that their everyday knowledge is what one thinks it to be. But there are other pitfalls. Voigt gives a minute and often disturbing account of a teacher trying to teach children a difficult and abstract mathematical concept through an everyday and relatively familiar example. His convincing description is of teacher and pupils trying to find the same context as each other and at first completely failing to do so. While the teacher wants them to think about the everyday example, they are still adopting a too formal mathematical approach. When he wants them to connect the everyday situation with mathematics they have swung too far in the other direction, and apparently decide that mathematics is not involved. Connections, Voigt shows, can be made between real life actions and classroom mathematics, but they are not easy to demonstrate.

Thus far I have shown that these papers confirm the gap. The separation between what children could in principle do and what they actually do is as large when it comes to mathematical proficiency as to other cognitive skills. Hatano's paper takes the argument a step further. He shows that Japanese children apply some, but not all, of the skills that they acquire when learning how to use the abacus to classroom mathematics. Moreover he gives a reason for this. It is that the workings of the abacus in more complex calculations are 'opaque'. The children trust the abacus, but they do not understand how it produces its answers to such problems and that prevents them from applying what they learn to complex sums that they have to do without the help of the abacus. On the other hand the abacus gives children a lot of practice with simple single figure computations and presumably helps them to automatize these calculations, which does in turn help them with similar sums in the classroom. There is transfer when the children can see the reason for it. The gap can be crossed.

Hatano also draws an analogy between the use of abacus by Japanese children and 'street mathematics' in Brazilian children—the phenomenon which was originally spotted and documented in a remarkably ingenious study by Carraher, Carraher and Schliemann (1985). 'Street mathematics' provide the most striking example of the gap that I have been discussing and of the importance of that gap. Carraher, Carraher and Schliemann showed that children selling fruit or other wares at their parents' market stalls were able to use a surprising range of mathematical moves, most of which they had probably discovered for themselves, in order to calculate the prices and the amount of change involved in different transactions. Yet these children often failed to use the same mathematical skills when given identical problems at school, even when the problems involved real life situations. Their mathematical skills seemed largely confined to the situation in which the skills had been acquired.

Hatano's reasons for connecting this phenomenon with learning to use the abacus are that both skills involve learning outside school and both are mainly used in commercial transactions. Of course the analogy is by no means perfect, because the abacus is often taught in a relatively formal way—as formal as in the classroom. Nevertheless Hatano's demonstration that there is some transfer from the abacus to the classroom raises the question whether there are any circumstances where Brazilian children's experiences with street mathematics could ever help them to learn arithmetic in the classroom.

The question seems even more pressing when one considers Saxe's interesting extension to our knowledge of street mathematics. His idea was to see how street traders dealt with rapid inflation which, as he points out, poses some quite difficult mathematical problems. Saxe's demonstration of the ingenuity and flexibility with which the traders surmount these problems is impressive, and of course it raises the question whether these people would have the same difficulty transferring their skills to other different situations which make the same mathematical demands. I think that it is safe to conclude that there would be very little transfer indeed.

I began these comments with the suggestion that these studies might provide some clue about the reason for the difficulty which children have in applying skills which they have learned in one situation to other situations where it would be just as appropriate to use these skills. In my view Hatano's study produces a real possibility. His argument is that children manage to transfer, when they have a clear rationale for what they are doing in different situations and thus can see more clearly that two different tasks though not the same on the surface nevertheless have the same underlying structure as each other.

One can put this argument in another way. Children make analogies between different problems when their understanding of the underlying structure of these problems is clear enough to let them see that both can be solved in the same way. It is not making analogies as such which is difficult for young children. The difficulty comes in seeing the connections between different problems. When children fail to apply street mathematics or what they have learned from the abacus to classroom problems, their difficulty is probably in seeing that there is a connection to make. Notice that the idea of analogies as a major cause of cognitive development is quite different from the ideas at the center of the two main causal models of cognitive development which I mentioned at the beginning of my discussion.

As it happens, there is some impressive recent research, outside the field of mathematical skills, to show that young children can make analogies when solving spatial problems (Brown, Kane and Echols, 1986) and also when they are learning to read (Goswami, 1986). Analogies are part of the child's life, and they are an excellent way—one could easily argue that they are the only way—to iron out the unevenness which leads children to do so well in one task and so poorly in another closely analogous one. Yet the evidence that children's progress in mathematics is held back by a persistent failure to make analogies which seem so clear to most adults is now very strong, and the papers in this issue, among other things, show how persistent this failure can be. I should like to suggest that the only possible conclusion to be drawn is that children make mathematical analogies when they realize that the underlying structure of two or more problems are the same despite the fact that their appearance on the surface is different. The surface differences which most distract them are almost certainly related to differences in the actions involved in different mathematical problems.

Our business is to work out how to help children to see the underlying similarities between, for example, problems which at first seem quite heterogeneous to them. This, surely, is an exciting task for them as well as for us.

References

Brown, A., Kane, M.J. & Echols, C.H. (1986) Young children's mental models determine analogical transfer across problems with common goal structure. *Cognitive Development*, 1, 103-121.

Bryant, P.E. (1985) The distinction between knowing when to do a sum and knowing how to do it. *Educational Psychology*, 5, 207-215.

Bryant, P.E., & Kopytynska, H. (1976) Spontaneous measurement by young children. *Nature*, 260, 773.

Carraher, T., & Bryant, P.E. (1987) Addition and subtraction as daily and mathematical concepts. *Proceedings of the China Satellite I.S.S.B.D. Conference*, Beijing, China.

Carraher, T., Carraher, D.W., & Schliemann, A. D. (1985) Mathematics in streets and schools. *British Journal of Developmental psychology*, 3, 21-30.

Cole M. (1985). The zone of proximal development: where culture and cognition create each other. In J.V. Wertsch (Ed.), *Culture, communication and cognition: Vygotskian perspectives*, 146-161. Cambridge: Cambridge University Press.

Fuson, K. (1988). *Children's counting and concept of number.* New York: Springer Verlag.

Goswami, U. (1986). Children's use of analogy in learning to read: A developmental study. *Journal of Experimental Child Psychology*, 42, 73-83.

Piaget, J. (1952). *The child's conception of number.* London: Routledge and Kegan Paul.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry.* London: Routledge & Kegan Paul.

Rose, S., & Blank, M. (1974). *The potency of context in children's cognition: an illustration through conservation. Child Development*, 45, 499-502.

Vygotsky, L. (1962). *Thought and language. Cambridge.* Mass.: M.I.T.Press

Vygotsky, L. (1978). *Mind in society.* Cambridge, Mass.:Harvard University Press.

The Pleasure and Pain of Mathematics: On Walkerdine's *The Mastery of Reason*

Derek Edwards
Loughborough University of Technology
Leicestershire, England

The Mastery of Reason (1988) is the latest of Valerie Walkerdine's explorations of the developmental and discursive origins of rationality. It explores a set of issues that arise out a close examination of how children talk about quantities, of what happens when they are inculcated ('inserted') into the adult-governed discursive practices of home and school, and what is going on when, citing Coghill (1978), she discovers a class of 5-year-olds who "booed the odd numbers and cheered the evens." Mathematics, like science and computers, for all their ostensible rationality and dispassionate objectivity, their general, context-free applicability, are nevertheless encountered as fearsome, pleasurable, powerful devices that can both fascinate and repel, reward and punish the learner, and especially, given their association with notions of power and control, may pose problems for the education and socialization of children for whom power and control are fraught values (girls, very often):

0278-4351/89/1&2-38 $1.00 © LCHC

> mastery of mathematics is not the end point of a naturally achieved maturation, or a developmental sequence which is universally human, as in all theories of cognitive development. Rather, it is a specific and powerfully created discourse in which power and control are inscribed in its very form (p.200).

Mathematics is the apogee of a view of knowledge that is embodied in Piaget and Freud, Darwin, science and logic—a "triumph of reason over emotion" (p. 5), in which the primary structuration of reason is held to be material reality.

Against this hegemony, Walkerdine poses a post-structural view of mathematics as cultural semiosis, a view derived from Foucault and Lacan, where mathematical rationality is encountered within a system of signs and meanings, discursive practices that are oriented to the construction of truth within the exercise of power, and to the regulation of the social order through science, psychology, medicine, and other such authoritative objectivities. It is this sort of perspective that informs, and is informed by, her explorations of children's talk.

It is currently fashionable to seek understandings of children's mental development, and in particular, their acquisition of rationality and mathematical competence, in terms of the contrasts and relationships that may obtain between their formal schooling, and their out of school understandings and activities. Walkerdine's approach is to pursue such issues not in terms of the development of cognitive skills, however these might be contextualized, but in terms of situated discursive practices. Mathematics and science are themselves discursive practices, with elements of fantasy, desire, and constraint: "the fantasy of discourse and practice in which the world becomes what is wanted: regular, ordered, controllable... with all aspects of value, emotionality, and desire suppressed" (p.188). To learn to use these discourses, children must learn to "forget" or "suppress" some richly meaningful content—not merely to acquire a neutrally objective and decontextualized mode of thought.

Walkerdine reveals in her investigation of classroom discourse many of the sorts of pedagogic procedures that Neil Mercer and I (Edwards and Mercer, 1987) have also discussed: the special and peculiar forms of discourse used in school tasks and classroom lessons: the importance of content rather than merely structure in examining pedagogic discourse (which, as Walkerdine notes, p. 89, is essential to the incorporation of an historical dimension into the analysis of discourse): the importance of discourse in establishing the significance of experience (rather than merely vice versa): and the problems which arise in the teacher's use of discursive devices which ostensibly elicit from children what is actually being taught to them. However, Walkerdine's study places its major focus on contrasts between the discourse of home and of school.

The empirical analysis centers on children's and adults' uses of what are usually taken to be proto-mathematical quantifiers in children's everyday speech: terms such as "more," "less," "a lot," "big," "little," and so on. But, rather than analyzing these as lexical descriptors which might be decomposed into a set of universal semantic features, or defined in terms of prototypical category definitions, Walkerdine studies their occurrence in situated discourse. She discovers them to be polysemic and functional, varying with pragmatic situational usage—such as the regulation of food portions, and general issues of food and drink consumption within family relationships, the distribution of limited resources, and such matters. The terms' referential appropriateness is shown to be conditional upon those contexts. Thus, "lot" and "little" were spontaneously used for contrasting food quantities, while "more" and "less" were not. Indeed, 'less' was not used at all in talk at home, except in what are termed "specifically pedagogic contexts."

So, while all of these quantificational words (and others) were part of children's vocabularies, and were used appropriately, their usage was demonstrably restricted to particular discursive practices, particular issues or domains of family relationships, rather than operating as universal quantifiers on absolute scales. Thus, the usual sets of abstracted opposites (more and less, big and little) that teachers, educated adults and analysts might expect to find, did not operate as such in the discursive practices within which the children were engaged at home.

With regard to the word "more," Walkerdine notes:

> It is striking that almost all the examples of *more* from this corpus form part of practices where the regulation of consumption is the object. In every case initiated by the child, she either wants more precious commodities, of which the mother sees it her duty to limit consumption, or the child does not want to finish food which the mother sees it her duty to make the child eat (p.26).

Walkerdine also argues that this word usage possesses a wide cultural-political significance:

> In terms of consumption within our culture, it is *more* which is valued... its value does not come from the internal relations of the linguistic system or a set of perceptual universals: it comes from the regulation of social practices which make up our culture... I would argue that every aspect of lexical development, for example, is amenable to such an analysis and moreover can be related to domestic, school, and work practice (p.27).

In school, a teacher is recorded telling her young class the story of "The Three Bears." It is not story time, but a mathematics lesson, on seriation and relative size. Instead of confusing them all with abstract talk of size and equivalence, numbers and arithmetic, she has chosen to employ the familiar story of the big Daddy bear, the middle-sized Mummy bear, and the small Baby bear. Some important principles of current developmental psychology are being applied: children learn abstract concepts best when these are introduced in terms of familiar experience and narratives. But the children still appear to be confused. It turns out that, in the ordinary discourse of life at home, Mummies are always "big," like Daddies, while children are mostly "little" (not "small"), but also "middle" or "big," depending on family discursive practices—one can be a little girl and a big sister and a "big girl now" and the one in the "middle," all in a short space of time. The terms "big" and "little" are deployed in highly significant discursive practices, where issues of socialization, identity, normality and deviance, praise and punishment, are at stake:

> Any girl's designation as 'big' or 'little' was not fixed, therefore, but depends on the practice and her position in it. The girl may be a 'little girl' but a 'big sister'... although the terms are used in a way which is not specific to size they are not used loosely, nor are they used indiscriminately, but indeed in very specific practices and circumstances to represent a specific set of relationships" (p.69)

Discursive practices at school, while often aimed at introducing formal concepts within friendly and familiar contexts, can easily result instead in "a complex and bewildering confusion" (p.47). Instead of helpful Donaldson-style (1978) human sense and "embeddedness," what we get is a clash between two discourses, with different significations—same signifiers, different signifiers, so different signs. The story invokes non-mathematical orders of everyday signification, to do with family life, authority, gender differences, etc., which are not transposable into the formal abstractions of mathematics. It is not that the children are doing mathematics, or protomathematics, outside of school, and need to learn a new context for it. They are doing something else with the words altogether. So:

> the problem for the children's insertion within school mathematics practices becomes one of prising apart and rearticulating the signifiers as entering into different relations of signification, in this case one in which...evaluative aspects of the signifier are suppressed (p.68).

In ordinary conversation, size-relation terms multiply signify all sorts of non-size practices and relationships between people, so that their use in the education of formal mathematics poses essentially discursive difficulties, and neither is it necessarily helped by the casual embedding of such terms in familiar narratives and "real-world" contexts.

Similarly, Walkerdine shows that in discourse at home, and in children's spontaneous narratives, money (a popular domain for school mathematics) is understood not merely as objects and quantities, but as involved in work-money-goods exchanges, subject to adult-child power issues, and the discourse of what adults cannot afford to buy for their children, issues of 'waste', and so on. Preschool children learn about money and value in the context of a *discourse of the domestic economy*—constraint, and money's purchasing power, of why adults have to go to work (why mummy or daddy can't be home yet, etc.), and what can and cannot be afforded, the necessity-luxury dichotomy, morally tinged with socializing issues of excess and waste. It is a social-moral nexus of relationships and power, significance and value. It is these sorts of significations that school mathematics must 'suppress', and in so doing, it suppresses the property of constructedness itself:

> this so-called natural process of mastery entails considerable and complex suppression. That suppression is both painful and extremely powerful. That power is pleasurable. It is the power of the triumph of reason over emotion, the fictional power over the practices of everyday life (p. 186).

Walkerdine's notion of discursive practice is one in which the material world clearly figures importantly, but in semiotic terms rather than 'objective' ones. Material reality is "slippery and mobile... only understood in terms of its meaningful insertion within particular discursive practices" (p.30). It is not immediately clear how discourse could have primacy over material reality, until one realizes what explanatory work is being done by the notion of semiosis. The material world is not the world-in-itself, but the world signified. This is a consideration that has implications for analysts as well as for participants:

> cooking... is commonly taken in nursery and infant schools [i.e., kindergarten] to be mathematics or pre-mathematics, if viewed as an opportunity for the concrete manipulation of certain objects. It is therefore instructive to consider that when a traditional developmental reading is imposed upon such a practice, it utilizes a quasi-mathematical discourse to do so. Logico-mathematical structures become a reading by which the psychologist or teacher *'sees'* mathematics or cognition in the activity in question. On the basis of this reading, the practice *'becomes'* cognition or mathematics (p.96-7: original *emphases*).

Semiotics studies all sorts of signification, not only language. Material reality and behavioral activity are important in so far as they signify, rather than just exist in the world. In this sense, "no practice is non-discursive" (p. 185). The notion that reality itself can be taken to be a discursive accomplishment effectively redefines the traditional distinctions between language, cognition and context, used as discrete categories in developmental psychology, but all of which become amenable to study on the same basis, which is the analysis of discourse. Further, it is a perspective which suggests the fruitfulness of looking for relationships between the construction of reality at home and in school on the one hand, and in the production both of ordinary common sense, and of scientific knowledge on the other.

While the latter concerns are the province of ethnomethodology, conversation analysis and the sociology of science (see, for example, Gilbert and Mulkay, 1984), Walkerdine appears to reject such approaches in favor of the French school of post-structural and psychoanalytical theory. Nevertheless, the sophistication of recent conversation-analytical, and discourse-analytical approaches to psychological and epistemological issues (e.g., Potter and Wetherell, 1987) goes well beyond "a simple view of turn-taking" where "there are not two equal participants" (Walkerdine, pp. 26 and 31), and indeed derives both from ethnomethodology, and also from the kind of post-modern epistemology of truth as discursive production that Walkerdine herself embraces. Her assertion that "if children are produced as subjects through their insertion as relations within specific practices, we should expect multiplicity and not singularity" (p.71) chimes closely with Potter and Wetherell's (op. cit.) discussion of the variability of accounts and the construction of 'self.' The time is ripe for an integrated discourse-analytical assault upon the general psychology of knowledge and of cognition.

References

Coghill, V. (1978). Infant school reasoning. Teachers' Research Group, unpublished mimeo. Cited in Walkerdine, 1988, p.68.

Donaldson, M. (1978). *Children's minds*. London: Fontana.

Edwards, D. & Mercer, N.M. (1987). *Common knowledge: The development of understanding in the classroom*. London: Methuen.

Gilbert, G.N. & Mulkay, M. (1984). *Opening pandora's box: A sociological analysis of scientists' discourse*. Cambridge: Cambridge University Press.

Potter, J. & Wetherell, M. (1987). *Discourse and social psychology: Beyond attitudes and behaviour*. London: Sage.

Walkerdine, V. (1988). *The mastery of reason: Cognitive Development and the Production of Rationality*. London: Routledge.

SUBMISSION OF MANUSCRIPTS: Please send three copies of manuscripts, double-spaced, figures and illustrations in **original, camera-ready form** to Peggy Bengel.

NOTICE OF SUBSCRIPTION RATE CHANGE: To help cut our loses we have increased our rates, effective January 1, 1988 to $20.00 per year; single and back issues are available for $5.00 each.

Subscription Form

Name ______________________________

Address ______________________________

______________________________ Zip ______________

Please enter my subscription to *The Quarterly Newsletter of the Laboratory of Comparative Human Cognition.*

I am enclosing $ for years at $20.00 per year

For mailing **outside the U.S. and Canada,** please add $7.00.

Please make your checks payable to UC Regents and mail them to:

Peggy Bengel
Laboratory of Comparative Human Cognition, X-003
University of California, San Diego
La Jolla, CA 92093

MOVING?

Please give us as much advance notice as possible and avoid missing an issue of the Newsletter.